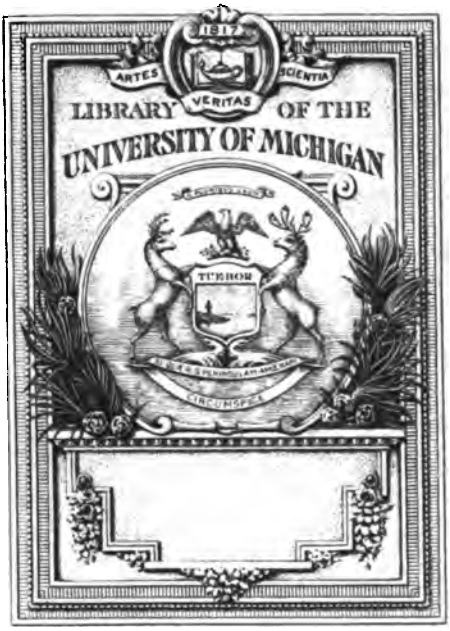
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ACTUARIAL THEORY

ACTUARIAL THEORY

NOTES FOR STUDENTS ON THE SUBJECT-MATTER REQUIRED IN THE SECOND EXAMINATIONS OF THE INSTITUTE OF ACTUARIES AND THE FACULTY OF ACTUARIES IN SCOTLAND, WITH NUMEROUS PRACTICAL EXAMPLES AND EXERCISES

BY WILLIAM A. ROBERTSON

Fellow of the Faculty of Actuaries in Scotland

AND

FREDERICK A. ROSS

Member of the Society of Accountants in Edinburgh; Fellow of the Faculty of Actuaries in Scotland

WITH A PREFATORY NOTE

BY THOMAS G. ACKLAND

Fellow of the Institute of Actuaries; Hon. Fellow of the Faculty of Actuaries in Scotland

OLIVER AND BOYD

EDINBURGH: TWEEDDALE COURT

LONDON: 10 PATERNOSTER ROW, E.C.

1907

PREFATORY NOTE

THE joint authors of this work consulted me, about two years since, as to the desirability of compiling and publishing a volume, much on the lines of *Graduated Exercises and Examples*, issued by Mr G. F. Hardy and myself in 1889, which work, owing to the material advance in actuarial science and in assurance practice since that date, has now become insufficient for the full needs of actuarial students. Being in entire agreement with the authors as to the demand for such a work, brought up to date, I encouraged them in their project, and now welcome the result of their labours.

The authors have kindly given me an opportunity of perusing a proof of this volume; and it is evident that they have devoted much care and labour to its production, and that their large and successful experience in training actuarial students has wisely guided them in the preparation of the work, which appears to me to form a most useful and illuminating commentary upon the admirable Institute Text Books.

A fairly large experience of actuarial students, both in their preliminary studies and in the examination room, has shown me two deficiencies frequently manifest in their work; first, the lack of original and independent thought, and a too slavish dependence upon the demonstrations and conclusions set out in the approved text books; and, secondly, a considerable failure in the power to apply, in practice, the results deduced theoretically; these two deficiencies being closely associated with one another. I have no doubt that the present work, by its elucidatory notes, alternative demonstrations, and illustrative examples (which deal not only with the fundamental bases of our Science, but also with its later practical developments), will prove most useful to students, by stimulating original thought and research, and thus enabling them to secure a firmer grip, both of the Theory and Practice of Actuarial Science.

THOMAS G. ACKLAND,
Fellow of the Institute of Actuaries,
Hon. Fellow of the Faculty of Actuaries in Scotland.

October 1907.

INTRODUCTION

Students preparing for the Second Examinations of the Institute of Actuaries and of the Faculty of Actuaries in Scotland have, to assist them at this stage of their studies, the *Text Book* of the Institute and Mr George King's *Theory of Finance*, combined with the *Graduated Exercises and Examples* of Messrs Ackland and Hardy. But there is good reason for believing that, with the extension of the purely actuarial part of the examinations, these works are no longer sufficient to enable even a careful student to take his examination with confidence. To supply a lack so important is therefore the intention of the authors in compiling this book for students; of whom even those preparing for the later examinations will find some parts of it not unworthy of study. As explained below, however, it is not a substitute for, but merely a supplement to, the works already mentioned, which, it must be urged, there is no intention to disparage in any way. Encouragement to proceed has come to the authors from various directions: from those whom they have had the privilege of assisting in their preparation for examinations, from their contemporaries in the profession, and, above all, from Mr Thomas G. Ackland, whose Prefatory Note they value very highly, as well as his kindly advice on many points.

Strictly speaking, the book is a compilation of notes on numerous points which are not disposed of in the text books so thoroughly as present-day exigencies require. No claim is made to originality, for that were futile: the matter consists of extracts from contributions to the *Journal* of the Institute and other professional records, or of explanations and elaborations of problems and statements contained in the text books. The effort throughout has been to simplify the obscure and to introduce only the essential.

In the authors' opinion, no student can hope to become proficient if he confines himself to reading the various books: it is necessary that he should deduce every formula for himself at least so often that he shall be confident that his own result will correspond with that of the text book, and confidence is essential in the working out of actuarial problems. It may be true that in such work memory is all important; the true use of memory, however, will be found, not in learning results by

heart, but rather in the application of the proper methods of deduction, and this will only come by practice.

An inspection of the contents of the book will show that it is based upon *The Theory of Finance* and the *Institute of Actuaries' Text Book*, Part II. These fundamental works must of course be read side by side with this; otherwise it will in great part lose its force. Ample references are made throughout to enable the student to follow with a minimum of trouble. The authors have been accustomed, both in studying and in teaching the subject of interest and annuities-certain, to the use of Mr King's book rather than the *Text Book*, Part I. But the student will find it advantageous also to follow closely the demonstrations and practical applications given by Mr Todhunter in the latter work.

No attempt has been made to deal with the purely mathematical side of the work. The three chapters at the close of the *Text Book*, Part II., and the subject of the calculus scarcely come within the scope of a work such as this.

The examples are taken for the most part from the examination papers of the Institute and the Faculty; and the answers, which follow immediately after the respective questions, have been prepared with care. The student should, of course, work answers to these and other examples independently, though not until the subject-matter of the books has been thoroughly grasped and mastered. It will frequently happen that the answer obtained by him will vary from that given; in which case it will be a useful exercise to prove the two identical, or, if they are clearly not so, to find where and how the difference arises. The authors will be glad if any errors which are discovered are pointed out to them.

It should be mentioned that, following the *Text Book*, they have preferred the more familiar $|_{t}q_{x}$ to the more officially correct $|_{t}Q_{x}$. Further, in the discussion of policy-values they have used the symbols ${}_{n:t}V_{x}$ and ${}_{n:t}U_{x}$ to represent the ordinary and special reserves after n years for whole-life policies with premiums limited to t years. Otherwise they conform to Institute notation.

Their grateful thanks are due to Mr John H. Imrie, M.A., F.F.A., and Mr Thomas Frazer, jun., F.F.A., who have read the proofs, and made many valuable suggestions.

W. A. ROBERTSON.
F. A. ROSS.

EDINBURGH, *October* 1907.

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ACTUARIAL THEORY

THEORY OF FINANCE

CHAPTER I

Interest

1. The first matter requiring attention is the question of the difference between the nominal and effective rates of interest.

In explanation of Article 12, it may be pointed out that, where a loan is made at 5 per cent. (for example), the interest is, in the ordinary case, payable half-yearly. Now the theory of compound interest is that interest earns interest, and therefore the interest paid at the end of six months earns interest to the end of the year. In this way the interest actually earned is over 5 per cent., though the loan is always nominally a 5 per cent. loan. The amount of a unit at the end of six months is 1·025, for the interest then paid is ·025. Starting then on the second six months with 1·025 of principal we have the interest thereon for the second six months 1·025 × ·025, and the amount of principal and interest at the end of that time will be

$$1{\cdot}025 \times 1{\cdot}025 = (1{\cdot}025)^2 = 1{\cdot}050625,$$

which is therefore the amount of 1 at the end of a year, and the actual interest on 1 for that period is ·050625 or £5, 1s. 3d. per cent. By similar reasoning, the general formula (14) follows:—

$$i^{(m)} = \left\{\left(1 + \frac{i}{m}\right)^m - 1\right\}$$

where i is the *nominal* rate of interest convertible m times a year, and $i^{(m)}$ is the corresponding *effective* rate of interest.

From the above we arrive at the following statements:—

The *Nominal Rate of Interest* is the rate per annum at which interest is quoted, no matter how often within the year that interest is convertible.

The *Effective Rate of Interest* is the total interest realised by the investment of a unit for a year.

Now interest may be convertible half-yearly, quarterly, monthly, or at the close of any fixed intervals. And these intervals may be reduced in length, until at last we have interest convertible at infinitely short periods, *i.e.*, momently. In this case, in formula (14), we write $\bar{i}$ for $i^{(m)}$ and δ for i; and we have $\bar{i} = \left\{ \left(1 + \frac{\delta}{m}\right)^m - 1 \right\}$ m being infinitely great. But by the theory of logarithms $\left(1 + \frac{\delta}{m}\right)^m$ in the limit becomes e^δ and we have

$$\bar{i} = e^\delta - 1$$

$$\text{and} \quad \delta = \log_e(1 + \bar{i})$$

δ is called the Force of Interest. It takes the place of the nominal rate of interest only when interest is convertible momently. We therefore define the *Force of Interest* as the nominal yearly rate of interest when interest is convertible momently, or the annual rate per unit at which a sum of money is increasing by interest at any moment of time.

2. To find the amount of 1 at the end of the pth part of a year where interest is convertible q times a year.

The amount of the unit at the end of a year will be $\left(1 + \frac{i}{q}\right)^q = (1 + i^{(q)})$. But by Article 16 the amount of 1 at the end of the pth part of a year at rate i is $(1+i)^{\frac{1}{p}}$, and therefore at rate $i^{(q)}$ is $(1 + i^{(q)})^{\frac{1}{p}} = \left(1 + \frac{i}{q}\right)^{\frac{q}{p}}$

3. Articles 17-21. Discount is defined as the difference between a sum due at the end of a given term and the present value thereof.

Discount assumes three forms according as it is calculated by the three following methods :—

(a) *Commercial Discount.*—In trade transactions, as in discounting a bill, the discount is calculated like simple interest at the quoted rate for the currency of the bill. That is to say, the discount is ni for each unit of the bill, n usually being fractional. If n were large, then the present value of the bill to be handed

over to the seller, $B(1 - ni)$, might be negative, which manifestly is absurd.

(b) *Simple Discount.*—The present value of a bill, B, due at the end of n years, where n may be fractional or integral, is $\frac{B}{1+ni}$, assuming simple interest. The discount, or by definition the difference between B and its present value, is therefore

$$B - \frac{B}{1+ni} = B\left(1 - \frac{1}{1+ni}\right)$$

(c) *Compound Discount.*—Again, the present value of a bill, B, due at the end of n years, where n as before may be fractional or integral, is by compound interest $\frac{B}{(1+i)^n}$, and the discount is accordingly, $B - \frac{B}{(1+i)^n} = B\left\{1 - \frac{1}{(1+i)^n}\right\}$

The formula for simple discount may be written in the form $ni\frac{B}{1+ni}$, and that for compound discount $\left\{(1+i)^n - 1\right\}\frac{B}{(1+i)^n}$, from both of which it will be seen that discount is really interest for the whole period on the present value of the sum, not, as is assumed in commercial discount, on the sum itself.

Discount may, in similar manner to interest, be convertible at any fixed intervals, and as is shown in Article 22, the value of 1 at the end of a year, where discount at nominal rate d is converted m times a year, is $\left(1 - \frac{d}{m}\right)^m$. Now, as before, the intervals may be made infinitely short, that is, discount may be convertible momently, and we have

$$\bar{v} = \left(1 - \frac{\delta}{m}\right)^m, \text{ where } \bar{v} \text{ is written for } v, \text{ and } \delta \text{ for } d.$$

In the limit when m is infinitely great $\left(1 - \frac{\delta}{m}\right)^m = e^{-\delta}$, whence

$$\bar{v} = e^{-\delta}$$

$$\text{and} \qquad -\delta = \log_e \bar{v}$$

$$= -\log_e(1+\bar{i})$$

Here, then, δ is called the *Force of Discount*. It is substituted for the nominal rate of discount when it is converted momently, and we may define it as the nominal rate of discount when

discount is converted momently, or the annual rate per unit at which a sum of money is decreasing by discount at any moment of time.

4. In Article 26 it is assumed, in finding the number of years in which money will double itself, that interest is convertible once a year. By a similar method it might be shown that, if interest were convertible m times a year, the number of $m$thly periods in which money would double itself would, by the first formula, be $\dfrac{\cdot 69}{\frac{i}{m}}$, and to find from this the number of years, it is necessary to divide by m, and we have therefore the number of years,

$$\frac{1}{m}\left\{\frac{\cdot 69}{\frac{i}{m}}\right\} = \frac{\cdot 69}{i}$$

the periods of conversion making no alteration in the length of time. It is obvious, however, that the length of time will be shorter the oftener interest is converted, and therefore it is necessary in this formula to use the effective rate of interest always. Thus:—

$$\left(1 + \frac{i}{m}\right)^{mn} = 2$$

$$\left[\left\{\left(1 + \frac{i}{m}\right)^{m} - 1\right\} + 1\right]^{n} = 2$$

$$(1 + i^{(m)})^{n} = 2$$

where $i^{(m)}$ is the effective rate corresponding to nominal rate i.

From this we get $n = \dfrac{\cdot 69}{i^{(m)}}$ approximately.

By the second and more exact formula, *i.e.*, $n = \dfrac{\cdot 693}{i} + \cdot 35$, on the other hand, the same error is not found. We have, if interest be convertible m times a year, the number of $m$thly periods $= \dfrac{\cdot 693}{\frac{i}{m}} + \cdot 35$. Dividing by m, as before, we get the number of years

$$= \frac{1}{m}\left\{\frac{\cdot 693}{\frac{i}{m}} + \cdot 35\right\} = \frac{\cdot 693}{i} + \frac{\cdot 35}{m}$$

If, however, we use the effective rate $i^{(m)}$ we have the number of years $= \frac{\cdot 693}{i^{(m)}} + \cdot 35$. These two formulas

$$n = \frac{\cdot 693}{i} + \frac{\cdot 35}{m}$$

$$\text{and} \qquad n = \frac{\cdot 693}{i^{(m)}} + \cdot 35$$

give results almost equal for ordinary rates of interest and periods of conversion, and this is a further proof of the superiority of the second formula over the first, as the nominal rate may be used without loss of accuracy. It is, however, necessary to note that the addition to be made to the result of dividing by the nominal rate is $\frac{1}{m}$ of $\cdot 35$ and not $\cdot 35$ as in the case where interest is payable yearly.

5. With regard to the equated time of payment, the proof that n, as found from

$$\frac{S_1 n_1 + S_2 n_2 + \cdot \cdot \cdot + S_r n_r}{S_1 + S_2 + \cdot \cdot \cdot + S_r}$$

is too great, is as follows :—

The Arithmetic Mean of S_1 quantities each v^{n_1} in amount, and S_2 quantities each v^{n_2} in amount, etc., and S_r quantities each v^{n_r} in amount, is equal to the total of the quantities divided by the number of quantities, or

$$\frac{S_1 v^{n_1} + S_2 v^{n_2} + \cdot \cdot \cdot + S_r v^{n_r}}{S_1 + S_2 + \cdot \cdot \cdot + S_r}$$

while their Geometric Mean is equal to the product of all the quantities to the root of the number of quantities, or

$$v^{\frac{S_1 n_1 + S_2 n_2 + \cdot \cdot \cdot + S_r n_r}{S_1 + S_2 + \cdot \cdot \cdot + S_r}}.$$

Now, as is shown below, the Arithmetic Mean of any set of quantities is greater than their Geometric Mean. Therefore

$$\frac{S_1 v^{n_1} + S_2 v^{n_2} + \cdot \cdot \cdot + S_r v^{n_r}}{S_1 + S_2 + \cdot \cdot \cdot + S_r} > v^{\frac{S_1 n_1 + S_2 n_2 + \cdot \cdot \cdot + S_r n_r}{S_1 + S_2 + \cdot \cdot \cdot + S_r}}$$

or $\quad (S_1 v^{n_1} + S_2 v^{n_2} + \cdot \cdot \cdot + S_r v^{n_r}) >$

$$(S_1 + S_2 + \cdot \cdot \cdot + S_r)\ v^{\frac{S_1 n_1 + S_2 n_2 + \cdot \cdot \cdot + S_r n_r}{S_1 + S_2 + \cdot \cdot \cdot + S_r}}$$

That is to say, the present value of S_1 due at the end of n_1 years, S_2 due at the end of n_2 years, etc., and S_r due at the end of n_r years, is greater than the present value of $(S_1+S_2+\cdot\cdot+S_r)$ due at the end of

$$\frac{S_1n_1+S_2n_2+\cdot\cdot\cdot+S_rn_r}{S_1+S_2+\cdot\cdot\cdot+S_r}$$

years (or n years). Therefore

$$\frac{S_1n_1+S_2n_2+\cdot\cdot\cdot+S_rn_r}{S_1+S_2+\cdot\cdot\cdot+S_r}\quad(\text{or } n)$$

is greater than the correct equated time of payment.

Proof that the Arithmetic Mean of n positive quantities is greater than their Geometric Mean.

The Arithmetic Mean of the n quantities $a, b, c, \cdot\cdot\cdot k$, is

$$\frac{a+b+c+\cdot\cdot\cdot+k}{n}$$

while their Geometric Mean is

$$(abc\ \cdot\cdot\cdot\ k)^{\frac{1}{n}}.$$

Now, in place of each of the greatest and least of these quantities, say a and k, put $\frac{a+k}{2}$. It may be easily proved that $\left(\frac{a+k}{2}\right)^2 > ak$, and therefore the result has been to increase the Geometric Mean while the Arithmetic Mean obviously remains as before, since

$$a+k=\frac{a+k}{2}+\frac{a+k}{2}$$

In place of each of the two quantities which are now the greatest and the least, put their Arithmetic Mean as before. The result is again to increase the Geometric Mean of the n quantities, while their Arithmetic Mean remains the same. This process may be repeated until the quantities are all, as nearly as possible, of equal value, in which case the Geometric Mean is equal to the Arithmetic Mean, for

$$(r.r.r\cdot\ \cdot\text{ to } n \text{ factors})^{\frac{1}{n}}=(r^n)^{\frac{1}{n}}=r=\frac{r+r+r+\cdot\ \cdot\text{ to } n\text{ terms}}{n}$$

But we have seen that the Arithmetic Mean remains the same throughout, while the Geometric Mean has been increased at each step until it equals the Arithmetic Mean. Consequently the first

Geometric Mean (of the n original quantities) must be less than the final Geometric Mean (of the n equalised quantities), that is, less than the Arithmetic Mean of the n original quantities, and we have

$$(abc \cdot \cdot \cdot k)^{\frac{1}{n}} < \frac{a+b+c+ \cdot \cdot \cdot +k}{n}.$$

EXAMPLES

1. A sum of £500 payable certainly at the end of 20 years is purchased for £239, 8s. 11d. Find the rate of interest realised by the investment.

Here we have

$$239{\cdot}446 = 500\,v^{20}$$

$$\text{Hence} \quad v^{20} = {\cdot}478892.$$

Resorting to the use of logs, we get

$$\log v = \frac{\log {\cdot}478892}{20}$$

$$\text{whence} \quad v = {\cdot}96386$$

$$\text{and} \quad i = {\cdot}0375.$$

The rate realised is therefore $3\frac{3}{4}$ per cent.

2. Verify the following figures:—

Nominal Rate.	Effective Rate, Interest being Convertible		
	Half-Yearly.	Quarterly.	Momently.
·04	·04040	·04060	·04081
·05	·05063	·05095	·05127

Effective Rate.	Nominal Rate, Interest being Convertible		
	Half-Yearly.	Quarterly.	Momently.
·035	·03470	·03455	·03440
·045	·04450	·04426	·04402

3. (*a*) What is the amount of £100 at the end of seven years, interest $4\frac{1}{2}$ per cent. convertible half-yearly? (*b*) What is the

present value of £250 due at the end of twelve years, interest 4 per cent. convertible quarterly?

Answers: (*a*) $£100 \times (1\cdot0225)^{14} = £136$, 11s. nearly.

(*b*) $£250 \times \dfrac{1}{(1\cdot01)^{48}} = £155$, 1s. 3d. nearly.

4. There are two sums of money, £A and £B, due at the end of n and m years respectively. (*a*) Find p the equated time of payment. (*b*) If this equated time of payment be extended to r years, to what sum will the amount due fall to be increased? Interest to be at rate i.

(*a*) By the approximate formula, we have

$$p = \frac{nA + mB}{A + B}$$

(*b*) Let C be the addition to be made to A + B if the time of payment be deferred to r.

$$\text{Then} \quad \frac{A+B+C}{(1+i)^r} = \frac{A}{(1+i)^n} + \frac{B}{(1+i)^m}$$

$$\text{and} \quad (A+B+C)(1-ri) = A(1-ni) + B(1-mi) \text{ approximately,}$$

$$\text{whence} \quad C = \frac{Ai(r-n) + Bi(r-m)}{1-ri} \text{ approximately.}$$

5. The premium income of an Insurance Office is distributed throughout the year as follows:—

Premiums due in	Amount.	Premiums due in	Amount.
January	£1000	July	£1900
February	£1100	August	£2000
March	£1250	September	£2300
April	£1500	October	£2800
May	£1700	November	£4000
June	£1850	December	£6000

Assuming that the premiums in each month are due on the average in the middle of the month, find the equated time of payment.

Here we have

$$n = \frac{(1000 \times \frac{1}{2}) + (1100 \times 1\frac{1}{2}) + (1250 \times 2\frac{1}{2}) + \cdot \cdot \cdot + (6000 \times 11\frac{1}{2})}{1000 + 1100 + 1250 + \cdot \cdot \cdot + 6000}$$

$$= 7 \cdot 768 \text{ months approximately.}$$

6. If a sum of money at a given rate of interest accumulate to p times its original amount in n years, and to p' times its original amount in n' years, show that $n' = n \log_p p'$.

By the terms of the question

$$\begin{aligned} (1+i)^n &= p \\ \text{whence} \quad n \log_p(1+i) &= \log_p p \\ &= 1 \\ \text{and} \quad n &= \frac{1}{\log_p(1+i)} \\ \text{Again} \quad (1+i)^{n'} &= p' \\ \text{Therefore} \quad n' \log_p(1+i) &= \log_p p' \\ \text{and} \quad n' &= \frac{\log_p p'}{\log_p(1+i)} \\ &= n \log_p p' \end{aligned}$$

7. A sum of money is invested at 3 per cent. Find approximately in how many years the sum will have increased to four times the original figure.

From G. F. Hardy's formula we may obtain the time within which the sum will double itself—

$$\begin{aligned} n &= \frac{\cdot 693}{\cdot 03} + \cdot 35 \\ &= 23 \cdot 45 \end{aligned}$$

The sum at the end of 23·45 years, being double that at the beginning, only needs to be redoubled to reach four times the original amount. The time required for this operation is, of course, again 23·45 years. Therefore the time required for money to become fourfold the original sum is 46·9 years.

CHAPTER II

Annuities-Certain

1. The amount of an annuity-due of 1 per annum for n years, interest at rate i convertible yearly, is as follows:—

$$\begin{aligned}&(1+i)+(1+i)^2+(1+i)^3+\cdot\cdot\cdot+(1+i)^n\\ =\ &(1+i)s_{\overline{n}|}\ \text{ or }\ s_{\overline{n+1}|}-1.\end{aligned}$$

The value of the same annuity-due is as follows:—

$$\begin{aligned}\mathrm{a}_{\overline{n}|}&=1+v+v^2+\cdot\cdot\cdot+v^{n-1}\\ &=(1+i)a_{\overline{n}|}\ \text{ or }\ 1+a_{\overline{n-1}|}\end{aligned}$$

2. The value of an annuity-due of 1 per annum for n years, payable p times a year, interest at rate i convertible q times, is found as follows:—

$$\mathrm{a}=\frac{1}{p}\left\{1+\left(1+\frac{i}{q}\right)^{-\frac{q}{p}}+\left(1+\frac{i}{q}\right)^{-\frac{2q}{p}}+\cdot\cdot+\left(1+\frac{i}{q}\right)^{-\frac{(np-1)q}{p}}\right\}$$

$$=\frac{1}{p}\frac{1-\left(1+\frac{i}{q}\right)^{-nq}}{1-\left(1+\frac{i}{q}\right)^{-\frac{q}{p}}}$$

$$=\left(1+\frac{i}{q}\right)^{\frac{q}{p}}\frac{1}{p}\frac{1-\left(1+\frac{i}{q}\right)^{-nq}}{\left(1+\frac{i}{q}\right)^{\frac{q}{p}}-1}$$

That is, the value of an annuity-due is equal to the value of an ordinary annuity of this nature with all the payments advanced by $\frac{1}{p}$ of a year, which is obviously correct.

3. Formula (13) may be written in the form—

$$a=\frac{1}{q}\frac{1-\left(1+\frac{i}{q}\right)^{-nq}}{\frac{i}{q}}$$

—from which it is more easily seen that an annuity for n years, both payable and with interest convertible q times a year, is the equivalent of an annuity of $\frac{1}{q}$ for nq periods, calculated at interest $\frac{i}{q}$, or we may write—

$$a_{\overline{n}|}^{(q)} = \frac{1}{q} a_{\overline{nq}|} \text{ at rate } \frac{i}{q}$$

If i be the effective rate of interest, we have :—

$$\begin{aligned} a_{\overline{n}|}^{(q)} &= \frac{1-(1+i)^{-n}}{q\{(1+i)^{\frac{1}{q}}-1\}} \\ &= \frac{1-(1+i)^{-n}}{i} \times \frac{i}{q\{(1+i)^{\frac{1}{q}}-1\}} \\ &= a_{\overline{n}|} \times \frac{i}{q\{(1+i)^{\frac{1}{q}}-1\}} \end{aligned}$$

The following explanation of this formula may be offered in supplement of that given in Article 26 :—If the q payments of $\frac{1}{q}$ each, payable at the end of each qth part of the year, were to be superseded by one payment at the end of the year, this single payment must be made equal to the accumulation to the end of the year of all the q payments of $\frac{1}{q}$ each ; that is, equal to

$$\frac{1}{q}\left\{1+(1+i)^{\frac{1}{q}}+(1+i)^{\frac{2}{q}}+\cdot\cdot\cdot+(1+i)^{\frac{q-1}{q}}\right\}$$

$$= \frac{1}{q}\frac{i}{(1+i)^{\frac{1}{q}}-1}$$

We therefore have the value of an annuity of 1 per annum, payable q times a year in instalments of $\frac{1}{q}$ each, equal to the value of an annuity of

$$\frac{i}{q\{(1+i)^{\frac{1}{q}}-1\}}$$

payable once a year, or in symbols

$$a_{\overline{n}|}^{(q)} = a_{\overline{n}|} \times \frac{i}{q\{(1+i)^{\frac{1}{q}} - 1\}}$$

In formula (13) as modified at the beginning of this section, if n be increased indefinitely the annuity is changed to a perpetuity, and the term involving n in the numerator disappears; we have

$$a_{\infty}^{(q)} = \frac{1}{q}\frac{1}{\frac{i}{q}}\left(= \frac{1}{q}\, a_{\infty} \text{ at rate } \frac{i}{q}\right) = \frac{1}{i}$$

as the value of a perpetuity at nominal rate of interest i, perpetuity payable and interest convertible q times a year.

The value of such a perpetuity may also be found as follows:—

The value of the first instalment is $\frac{1}{q}\left(1 + \frac{i}{q}\right)^{-1}$, of the second $\frac{1}{q}\left(1 + \frac{i}{q}\right)^{-2}$, and so on. We therefore have

$$\begin{aligned}
a_{\infty}^{(q)} &= \frac{1}{q}\left\{\left(1 + \frac{i}{q}\right)^{-1} + \left(1 + \frac{i}{q}\right)^{-2} + \left(1 + \frac{i}{q}\right)^{-3} + \cdot \cdot \textit{ ad inf.}\right\} \\
&= \frac{1}{q} \times \frac{\left(1 + \frac{i}{q}\right)^{-1}}{1 - \left(1 + \frac{i}{q}\right)^{-1}} \\
&= \frac{1}{q} \times \frac{1}{\left(1 + \frac{i}{q}\right) - 1} \\
&= \frac{1}{q}\frac{1}{\frac{i}{q}} \\
&= \frac{1}{i}
\end{aligned}$$

That this value is correct may be shown thus:—If 1 be invested at this rate of interest it will provide $\frac{i}{q}$ at the end of each $\frac{1}{q}$ of a year (which is the same thing as saying that, if it be invested at rate $\frac{i}{q}$ per $q$thly period, it will yield $\frac{i}{q}$ at the

end of each such period), and therefore, if $\frac{1}{i}$ be invested, by simple proportion it will yield $\frac{1}{q}$ at the end of each period, which is the perpetuity we desire.

Similarly in formula (13a), let $p=q$ and let n be increased indefinitely, and the annuity becomes a perpetuity, the term involving n vanishing as before. We then have

$$a_{\infty}^{(q)} = \frac{1}{q}\,\frac{1}{\{(1+i^{(q)})^{\frac{1}{q}}-1\}}$$

where $i^{(q)}$ is the effective rate of interest convertible q times a year.

This perpetuity may also be valued thus:—The value of the first instalment is $\frac{1}{q}(1+i^{(q)})^{-\frac{1}{q}}$, of the second $\frac{1}{q}(1+i^{(q)})^{-\frac{2}{q}}$, and so on. We therefore have

$$a_{\infty}^{(q)} = \frac{1}{q}\left\{(1+i^{(q)})^{-\frac{1}{q}}+(1+i^{(q)})^{-\frac{2}{q}}+(1+i^{(q)})^{-\frac{3}{q}}+\cdots\ ad\ inf.\right\}$$

$$= \frac{1}{q}\times\frac{(1+i^{(q)})^{-\frac{1}{q}}}{1-(1+i^{(q)})^{-\frac{1}{q}}}$$

$$= \frac{1}{q}\,\frac{1}{(1+i^{(q)})^{\frac{1}{q}}-1}$$

This, then, is the value of a perpetuity at effective rate of interest $i^{(q)}$, perpetuity payable and interest convertible q times a year, and may be explained thus:—If 1 be invested at this rate of interest, it will yield at the end of each $\frac{1}{q}$ of a year $\{(1+i^{(q)})^{\frac{1}{q}}-1\}$, and therefore $\frac{1}{\{(1+i^{(q)})^{\frac{1}{q}}-1\}}$ will produce 1 at the end of each such period, and

$$\frac{1}{q}\,\frac{1}{\{(1+i^{(q)})^{\frac{1}{q}}-1\}}$$

will produce $\frac{1}{q}$ at the end of each period, which is the perpetuity required.

4. The following equations should be carefully noted:—

$$
\begin{aligned}
(1+i)^{m+n} &= (1+i)^m \times (1+i)^n \\
v^{m+n} &= v^m \times v^n \\
s_{\overline{m+n}|} &= s_{\overline{m}|} + (1+i)^m s_{\overline{n}|} \\
a_{\overline{m+n}|} &= a_{\overline{m}|} + v^m a_{\overline{n}|}
\end{aligned}
$$

These formulas are of importance in connection with Interest Tables. It may be desired to obtain the value in respect of $(m+n)$ intervals, where the values in the tables are tabulated in respect of intervals up to m only.

5. The value of future fines for the renewal of a lease, or, in Scotland, of future duplicands of feu-duty, and the substitution for them of an equal annual payment in perpetuity may be considered as follows:—

Suppose F the duplicand due now and at the end of every tth year from now. Then the present value of all the payments of F in perpetuity is

$$
\begin{aligned}
& F(1+v^t+v^{2t}+v^{3t}+ \cdot \cdot \cdot \textit{ ad inf.}) \\
&= F\frac{1}{1-v^t}
\end{aligned}
$$

Now let P be the annual payment to be found which will be substituted for the periodical payments of F.

Then the present value of all the payments of P, assuming the first to be due now, is

$$
\begin{aligned}
& P(1+v+v^2+v^3+ \cdot \cdot \cdot \textit{ ad inf.}) \\
&= P\frac{1}{1-v} = P\frac{1+i}{i}
\end{aligned}
$$

Now the present values of these two series of payments must be equal to one another, and we therefore have

$$
\begin{aligned}
P\frac{1}{1-v} &= F\frac{1}{1-v^t} \\
\text{whence} \quad P &= F\frac{1-v}{1-v^t} \\
&= F\frac{d}{1-v^t}
\end{aligned}
$$

But we see that the annual sum payable in advance for t years which a sum of F payable now will purchase is

$$F\frac{1}{(1+i)a_{\overline{t|}}} = F\frac{i}{1+i}\frac{1}{1-v^t} = F\frac{d}{1-v^t}$$

Our result is thus confirmed by general reasoning.

Suppose now the first payment of the duplicand be due t years hence; the present value of all the payments is then

$$\begin{aligned} &F(v^t+v^{2t}+v^{3t}+ \cdot \cdot \cdot \textit{ad inf.}) \\ =\ &F\frac{v^t}{1-v^t} \\ =\ &F\frac{1}{(1+i)^t-1} \end{aligned}$$

And the present value of all the annual payments of P, the first being assumed to be payable a year hence, is

$$\begin{aligned} &P(v+v^2+v^3+ \cdot \cdot \cdot \textit{ad inf.}) \\ =\ &P\frac{1}{i} \end{aligned}$$

We therefore have in a similar way as before

$$P\frac{1}{i} = F\frac{1}{(1+i)^t-1}$$

$$P = F\frac{i}{(1+i)^t-1}$$

Now the annual payment which requires to be set aside to accumulate to the sum F due at the end of t years is

$$F\frac{1}{s_{\overline{t|}}} = F\frac{i}{(1+i)^t-1}$$

this result also being arrived at by general reasoning.

6. The schedule given in Article 39 illustrating the redemption of a sum by equal payments including principal and interest is very instructive. It is shown how the capital contained in the mth payment of the annuity is $\frac{K}{a_{\overline{n|}}}v^{n-m+1}$. We also know that the capital contained in the first payment is $\frac{K}{s_{\overline{n|}}}$, and in the

second is $\frac{K}{s_{\overline{n|}}}(1+i)$, because the interest on the first repayment of capital has been released and must be utilised to increase the capital contained in the second payment. Similarly in the third instalment the capital is $\frac{K}{s_{\overline{n|}}}(1+i)^2$, and generally in the mth instalment the capital is $\frac{K}{s_{\overline{n|}}}(1+i)^{m-1}$.

That $\frac{K}{s_{\overline{n|}}}(1+i)^{m-1} = \frac{K}{a_{\overline{n|}}}v^{n-m+1}$ is easily proved.

For
$$\frac{K}{s_{\overline{n|}}}(1+i)^{m-1} = \frac{Kv^n(1+i)^{m-1}}{v^n s_{\overline{n|}}}$$
$$= \frac{Kv^{n-m+1}}{a_{\overline{n|}}}$$

By the first way of looking at the matter, the repayments of capital in t years amount to

$$\frac{K}{a_{\overline{n|}}}(v^n + v^{n-1} + \ldots + v^{n-t+1}) = \frac{K}{a_{\overline{n|}}}(a_{\overline{n|}} - a_{\overline{n-t|}}) = K - \frac{K}{a_{\overline{n|}}}a_{\overline{n-t|}}$$

And by the second the total capital repaid in t years is

$$\frac{K}{s_{\overline{n|}}}\{1+(1+i)+\cdots+(1+i)^{t-1}\} = \frac{K}{s_{\overline{n|}}}s_{\overline{t|}}$$

These two expressions are identical, for

$$\frac{K}{s_{\overline{n|}}}s_{\overline{t|}} = \frac{K}{v^n s_{\overline{n|}}}v^n s_{\overline{t|}} = \frac{K}{a_{\overline{n|}}}v^n\{1+(1+i)+\cdots+(1+i)^{t-1}\}$$
$$= \frac{K}{a_{\overline{n|}}}(v^n + v^{n-1} + \ldots + v^{n-t+1})$$
$$= \frac{K}{a_{\overline{n|}}}(a_{\overline{n|}} - a_{\overline{n-t|}})$$
$$= K - \frac{K}{a_{\overline{n|}}}a_{\overline{n-t|}}$$

Again, the capital returned in the first payment is $\frac{K}{a_{\overline{n|}}}v^n$, in the second $\frac{K}{a_{\overline{n|}}}v^{n-1}$, and so on, and in the last it is $\frac{K}{a_{\overline{n|}}}v$. Now

the present value of the capital in the first payment is $v\frac{K}{a_{\overline{n}|}}v^n = \frac{K}{a_{\overline{n}|}}v^{n+1}$, of that in the second $v^2\frac{K}{a_{\overline{n}|}}v^{n-1} = \frac{K}{a_{\overline{n}|}}v^{n+1}$, of that in the last $v^n\frac{K}{a_{\overline{n}|}}v = \frac{K}{a_{\overline{n}|}}v^{n+1}$. Therefore the total value of all the capital repaid in the n instalments is $n\frac{K}{a_{\overline{n}|}}v^{n+1}$.

This expression is of use in ascertaining the value to be paid for an annuity-certain allowing for income-tax, when tax is deducted from the whole annual payments without regard being had to the proportions of capital contained therein. It is obvious that, for an annuity of 1 for n years, a purchaser in these circumstances should not pay $a_{\overline{n}|}$, but should deduct the value of income-tax on capital at t per unit, or tnv^{n+1}. Thus the net price paid for the annuity will be $a_{\overline{n}|} - tnv^{n+1}$. This result is necessarily only approximate, as an adjustment should now be made for the reduction of interest following on the reduction of capital invested, and for the consequent increase of capital returned in the successive payments of the annuity.

In making up a schedule such as that given in Article 39, it should be carefully noticed that it is only necessary to work out the figures in column (3). The first value in this column is $\frac{K}{s_{\overline{n}|}}$. The succeeding values are obtained by continued multiplication by $(1+i)$. The figures in all the other columns are obtained from those in column (3). In forming the schedule in this way, however, a periodical check should be applied, the figure in column (3) opposite m being $\frac{K}{s_{\overline{n}|}}(1+i)^{m-1}$.

When the annuity is payable q times a year, it should be assumed that interest is convertible at the periods of payment of the annuity. The schedule should then be formed in respect of an annuity for nq intervals at rate of interest $\frac{i}{q}$, bearing in mind the formula previously found, namely

$$a_{\overline{n}|}^{(q)} = \frac{1}{q}a_{\overline{nq}|} \text{ at rate of interest } \frac{i}{q}$$

7. In the circumstances mentioned in Article 40, where an investor has lent money, repayable by an annuity and yielding a given rate of interest, say i, but where he is able to accumulate the sinking fund returned to him annually at a lower rate only, say i', it will be seen that for an advance of 1, the borrower must pay interest amounting to i per annum, and also the sinking fund at rate i' to replace the advance of 1 or $\frac{1}{s'_{\overline{n}|}}$. In other words, 1 is the value of an annuity of $i+\frac{1}{s'_{\overline{n}|}}$, and by proportion $\frac{s'_{\overline{n}|}}{1+is'_{\overline{n}|}}$ is the value of an annuity of 1 per annum.

To find in such a case the amount of capital outstanding at the end of t years.

If K was the original advance, the annual payment being

$$K\left(i + \frac{1}{s'_{\overline{n}|}}\right)$$

the sinking fund will have accumulated at i' to

$$\frac{K}{s'_{\overline{n}|}}\, s'_{\overline{t}|}$$

in t years. Now, if the borrower be asked to repay the capital outstanding for the convenience of the lender, he should pay only the balance outstanding after deduction of the accumulation of sinking fund, that is, $K - \frac{K}{s'_{\overline{n}|}}\, s'_{\overline{t}|}$.

If, on the contrary, it be to the borrower's convenience that he should repay the balance of capital, the lender must receive such a sum as will enable him to purchase an annuity of $K\left(i + \frac{1}{s'_{\overline{n}|}}\right)$ for the remainder of the term, that is, $K\left(i + \frac{1}{s'_{\overline{n}|}}\right)a'_{\overline{n-t}|}$, the value of the annuity being calculated at rate i', as that is the rate returned by investments elsewhere.

The third case may, however, arise where both parties desire to end the contract, and in such circumstances it will be sufficient if the lender get such a sum as will enable him to set up a similar

contract for the remainder of the term. That is, he should get the value of an annuity of $K\left(i + \frac{1}{s'_{\overline{n}|}}\right)$ for the unexpired period on the same terms as the original annuity was calculated. We saw that the value of an annuity of 1 for the whole n years was $\frac{s'_{\overline{n}|}}{1+is'_{\overline{n}|}}$. Hence we get the value of an annuity of $K\left(i + \frac{1}{s'_{\overline{n}|}}\right)$ for $(n-t)$ years as $K\left(i + \frac{1}{s'_{\overline{n}|}}\right)\frac{s'_{\overline{n-t}|}}{1+is'_{\overline{n-t}|}}$.

8. The general rule given in Article 43*a* for finding the present value of a series of payments of any amounts, to be made at any times, the value to be so calculated as to yield the purchaser the remunerative rate, i, on his whole investment throughout the longest of the periods, n years, and to return him his capital at the accumulative rate, i', should be most carefully noted, as it is invaluable in finding the present value of varying annuities of this nature. It is sufficient to know the first part of the rule; namely, that the present value may be found by multiplying the amount accumulated at rate i' to the end of the n years of the series of payments by $\frac{1}{1+is'_{\overline{n}|}}$.

9. In Article 50 it is shown how to approximate to the rate of interest by means of Finite Differences, given the value of the annuity and the term. By the same means an approximation may be made to the value of an annuity at a rate intermediate between the rates in a given table of values.

The general formula may be stated in the form

$$u_{nx+h} = u_{nx} + \frac{h}{x}\Delta u_{nx} + \frac{\frac{h}{x}\left(\frac{h}{x}-1\right)}{\underline{|2}}\Delta^2 u_{nx} + \cdot \cdot$$

where the values at intervals of x in the rate of interest are given.

For example, if tables of values at 3 per cent., $3\frac{1}{2}$ per cent., 4 per cent., etc., be given, and it is desired to find the value at, say, $3\frac{3}{4}$ per cent, we have

$$a_{(3\frac{3}{4}\%)} = a_{(3\%)} + \tfrac{3}{2}\Delta a_{(3\%)} + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2}\Delta^2 a_{(3\%)}$$

For a term of 20 years,

$$\begin{aligned} a_{(3\frac{3}{8}\%)} &= 14{\cdot}87748 + \tfrac{3}{2}(-{\cdot}66508) + \tfrac{3}{8} \times {\cdot}04301 \\ &= 14{\cdot}87748 - {\cdot}99762 + {\cdot}01613 \\ &= 13{\cdot}89599 \end{aligned}$$

10. With reference to Article 65, *a*, it should be noted that the value of an annuity-certain for n years deferred t years, interest at rate i during the first t years and at rate j thereafter, can be conveniently expressed only in the form

$$(1+i)^{-t}\,\frac{1-(1+j)^{-n}}{j}.$$

It should *not* be written in any modification of the formula

$$_{t|}a_{\overline{n}|} = a_{\overline{n+t}|} - a_{\overline{t}|}$$

11. To find the annual premium payable in advance for t years required to provide an annuity-certain for n years, the first payment of which is to be made at the end of t years.

The value of the benefit to be obtained is $v^{t-1}a_{\overline{n}|}$.

The value of the payments to be made to secure this benefit (P being the required annual premium) is

$$P(1+v+v^2+\cdots+v^{t-1}) = P\,\mathrm{a}_{\overline{t}|}$$

Now the value of the benefit must equal the value of the payments made for it, whence we have

$$\begin{aligned} P\,\mathrm{a}_{\overline{t}|} &= v^{t-1}a_{\overline{n}|} \\ \text{and}\quad P &= \frac{v^{t-1}a_{\overline{n}|}}{\mathrm{a}_{\overline{t}|}} \\ &= \frac{v^t\,a_{\overline{n}|}}{a_{\overline{t}|}} \\ &= \frac{a_{\overline{n}|}}{s_{\overline{t}|}} \end{aligned}$$

If the premium be payable half-yearly, we have as before the benefit side $= v^{t-1}a_{\overline{n}|}$, and the payment side

$$= \frac{P}{2}\left\{1 + \left(1+\frac{i}{2}\right)^{-1} + \left(1+\frac{i}{2}\right)^{-2} + \cdots + \left(1+\frac{i}{2}\right)^{-(2t-1)}\right\}$$

$$= \frac{P}{2}\mathrm{a}_{\overline{2t}|} \text{ where interest is at rate } \frac{i}{2}.$$

Therefore $$\frac{P}{2} = \frac{v^{t-1} a_{\overline{n}|}}{1 + a_{\overline{2t-1}|}}$$

$a_{\overline{2t-1}|}$ being calculated at rate $\frac{i}{2}$ and the other functions at rate i.

12. Sinking-Fund Assurances are of importance, as they are more frequently in use than formerly was the case. They are employed to provide sums required for the redemption of debenture issues at their due date, to return at the expiry of a lease the capital sum paid for property held on leasehold, and, in short, to secure the payment of a sum of whatever nature at the end of a term certain.

The present value of such a sum, that is, the single premium to secure it, is v^n.

Putting $P_{\overline{n}|}$ for the annual premium to secure this benefit, the present value of the premiums is

$$P_{\overline{n}|}(1 + v + v^2 + \cdot \cdot \cdot + v^{n-1}) = P_{\overline{n}|}(1 + a_{\overline{n-1}|})$$

whence $$P_{\overline{n}|}(1 + a_{\overline{n-1}|}) = v^n$$

and $$P_{\overline{n}|} = \frac{v^n}{1 + a_{\overline{n-1}|}}$$

If the premium be payable p times a year, we have the payment side equal to

$$\frac{P^{(p)}_{\overline{n}|}}{p}\left\{1 + \left(1 + \frac{i}{p}\right)^{-1} + \left(1 + \frac{i}{p}\right)^{-2} + \cdot \cdot + \left(1 + \frac{i}{p}\right)^{-(p-1)} + \cdot \cdot + \left(1 + \frac{i}{p}\right)^{-(np-1)}\right\} = \frac{P^{(p)}_{\overline{n}|}}{p}(1 + a_{\overline{np-1}|})$$

where $a_{\overline{np-1}|}$ is calculated at rate $\frac{i}{p}$.

From this we get

$$\frac{P^{(p)}_{\overline{n}|}}{p} = \frac{v^n}{1 + a_{\overline{np-1}|}}$$

We have so far made but a simple application of the formulas already obtained, interest being assumed to remain constant throughout the whole term of n years. It is, however, the case that the rate of interest has shown a tendency to decline for many

years, though of late this tendency appears to have received a check, which however is probably of only temporary effect. It will in any event be prudent to make allowance for such a fall, and we must seek formulas to give effect to this consideration.

Suppose the rate of $3\frac{1}{2}$ per cent. to hold for 10 years, thereafter falling $\frac{1}{4}$ per cent. every 10 years till a minimum of 2 per cent. is reached. Then the value of 1 due at the end of $(10+m)$ years $(m < 10)$ is $v^{10}_{(3\frac{1}{2})} \times v^{m}_{(3\frac{1}{4})}$.

At the end of $(20+m)$ years $(m < 10)$, the value is

$$v^{10}_{(3\frac{1}{2})} \times v^{10}_{(3\frac{1}{4})} \times v^{m}_{(3)}.$$

At the end of $(60+m)$ years the value is

$$v^{10}_{(3\frac{1}{2})} \times v^{10}_{(3\frac{1}{4})} \times v^{10}_{(3)} \times v^{10}_{(2\frac{3}{4})} \times v^{10}_{(2\frac{1}{2})} \times v^{10}_{(2\frac{1}{4})} \times v^{m}_{(2)}$$

where m has any value.

The present value of the annual premiums where the sum is due at the end of $(10+m)$ years $(m < 10)$, is

$$P\left(a_{\overline{10}|\,(3\frac{1}{2})} + v^{10}_{(3\frac{1}{2})}\, a_{\overline{m}|\,(3\frac{1}{4})}\right)$$

$$\text{and} \quad P = \frac{v^{10}_{(3\frac{1}{2})} \times v^{m}_{(3\frac{1}{4})}}{a_{\overline{10}|\,(3\frac{1}{2})} + v^{10}_{(3\frac{1}{2})}\, a_{\overline{m}|\,(3\frac{1}{4})}}$$

Similar values of P where the sum is due at the end of $(20+m)$, $(30+m)$, $(40+m)$, and $(50+m)$ years, m in each case being less than 10, may be found.

Finally, when the sum is due at the end of $(60+m)$ years, m being of any value, we have for the value of the annual premiums

$$P\left(a_{\overline{10}|\,(3\frac{1}{2})} + v^{10}_{(3\frac{1}{2})}\, a_{\overline{10}|\,(3\frac{1}{4})} + v^{10}_{(3\frac{1}{2})}\, v^{10}_{(3\frac{1}{4})}\, a_{\overline{10}|\,(3)} + \cdot \cdot \right.$$
$$\left. + \, v^{10}_{(3\frac{1}{2})}\, v^{10}_{(3\frac{1}{4})}\, v^{10}_{(3)}\, v^{10}_{(2\frac{3}{4})}\, v^{10}_{(2\frac{1}{2})}\, v^{10}_{(2\frac{1}{4})}\, a_{\overline{m}|\,(2)}\right).$$

And, the value of the benefit being as found above, we may at once determine the value of P.

18. To find the value of an annuity-certain of 1 for n years paying the purchaser a desired rate of interest and securing by a Sinking Fund policy the return of his capital with one year's interest at the end of the year following the last payment of the annuity.

A purchaser would pay 1 for an annuity-due for $(n+1)$ years of $(P_{\overline{n+1}|} + d)$, where $P_{\overline{n+1}|}$ is the premium payable in advance

charged by an office for a Sinking Fund policy of $(n+1)$ years term, and d is the interest in advance on 1 at the rate desired. For an annuity for n years of $(P_{\overline{n+1|}} + d)$ he would therefore pay $1-(P_{\overline{n+1|}} + d)$, and for an annuity of 1 for n years he would pay

$$\frac{1}{P_{\overline{n+1|}} + d} - 1.$$

Again, for the annuity of $(P_{\overline{n+1|}} + d)$ for n years, we saw that the policy effected was for 1, and therefore for an annuity of 1 the policy will be for $\frac{1}{P_{\overline{n+1|}} + d}$, and the annual premium will be

$$\frac{P_{\overline{n+1|}}}{P_{\overline{n+1|}} + d}.$$

We have now to see

(1) What the total capital invested is;

(2) How each annual payment is divided between interest and premium; and

(3) Whether the policy returns the capital invested with one year's interest at the end of $(n+1)$ years.

(1) The value paid for the annuity is, as above, .	$\frac{1}{P_{\overline{n+1|}}+d} - 1$
But in addition the purchaser must pay the first premium on the Sinking Fund policy, which is	$\frac{P_{\overline{n+1|}}}{P_{\overline{n+1|}}+d}$
Therefore the total capital invested is . .	$\frac{v}{P_{\overline{n+1|}}+d}$

(2) Each annual payment is 1,

whereof there is

Interest on $\frac{v}{P_{\overline{n+1|}}+d}$ of capital	$\frac{d}{P_{\overline{n+1|}}+d}$
And premium on policy	$\frac{P_{\overline{n+1|}}}{P_{\overline{n+1|}}+d}$
Together	1

(3) The capital invested is, as before, . . . $\dfrac{v}{P_{\overline{n+1}|}+d}$

One year's interest thereon . . . . . $\dfrac{d}{P_{\overline{n+1}|}+d}$

Together, making up the amount payable under the policy . . . . . $\dfrac{1}{P_{\overline{n+1}|}+d}$

If $P_{\overline{n+1}|}$ be a net premium, and calculated at the same rate as d, then the price paid for the annuity, $\dfrac{1}{P_{\overline{n+1}|}+d}-1$, is equal to $a_{\overline{n}|}$, the value of an annuity-certain for n years.

For, since
$$P_{\overline{n+1}|} = \frac{1}{(1+i)s_{\overline{n+1}|}} = \frac{1}{1+i}\left(\frac{1}{a_{\overline{n+1}|}} - i\right)$$
$$= \frac{1}{\text{a}_{\overline{n+1}|}} - d$$
$$P_{\overline{n+1}|} + d = \frac{1}{\text{a}_{\overline{n+1}|}}$$
$$\frac{1}{P_{\overline{n+1}|}+d} = \text{a}_{\overline{n+1}|}$$

And
$$\frac{1}{P_{\overline{n+1}|}+d} - 1 = \text{a}_{\overline{n+1}|} - 1$$
$$= a_{\overline{n}|}.$$

EXAMPLES

1. If an annuity-certain is payable twice a year, interest convertible four times a year, and the effective rate of interest is i, what is the amount of the annuity in n years?

The general formula to be applied is

$$s = \frac{1}{p}\,\frac{\left(1+\frac{j}{q}\right)^{nq}-1}{\left(1+\frac{j}{q}\right)^{\frac{q}{p}}-1}$$

where j is the nominal rate of interest convertible q times a year. But by the terms of the question $\left(1+\frac{j}{q}\right)^q = (1+i)$ where $q=4$. Also $p=2$. Therefore we have

$$s = \frac{1}{2}\frac{(1+i)^n - 1}{(1+i)^{\frac{1}{2}} - 1}$$

2. Find the amount per annum payable momently, interest convertible momently, for n years, corresponding to a yearly payment of a for n years, interest convertible yearly.

Let K be the amount per annum required. Then the value of K for n years will be $K\frac{1-e^{-n\delta}}{\delta}$, which must be equated to the value of the payments of a, that is, to $a\frac{1-(1+i)^{-n}}{i}$.

We therefore have

$$K\frac{1-e^{-n\delta}}{\delta} = a\frac{1-(1+i)^{-n}}{i}$$

$$\text{whence} \quad K = a\frac{1-(1+i)^{-n}}{i} \times \frac{\delta}{1-e^{-n\delta}}$$

3. An annuity-due of 1 per annum is to be allowed to accumulate until the first payment has doubled itself. Assuming that this occurs at the end of an integral number of years exactly, find what is then the amount of the annuity. Prove the result by general reasoning.

If n be the number of years it takes the first payment to double itself, we have the amount of the annuity-due at the end of that time equal to

$$(1+i)+(1+i)^2+ \cdots +(1+i)^n$$

$$= (1+i)\frac{(1+i)^n - 1}{i}$$

$$= (1+i)\frac{2-1}{i}$$

since $(1+i)^n = 2$

$$= \frac{1+i}{i}$$

This is the value of a perpetuity-due of 1 per annum, and our result is easily proved to be correct. For, the first payment having accumulated to 2, of this 1 may be paid away and the remaining 1 accumulated for n years further, while the second and succeeding payments will accumulate to 2 in succession, yielding 1 per annum

to be paid away and 1 per annum to be re-invested and accumulated, and so on, *ad infinitum*; all which is obviously the value of a perpetuity-due.

4. Find the value of an annuity-certain of 1 payable half-yearly for 48 years and 48 days at £3, 3s. 2d. per cent. interest, given log 1·01579 = ·006804 and log ·22132 = $\bar{1}$·345027.

We must assume here that interest is convertible half-yearly, and then remembering that where both annuity is payable and interest convertible p times a year $a^{(p)}_{\overline{n|}} = \frac{1}{p} a_{\overline{np|}}$ at rate of interest $\frac{i}{p}$, we have

$$a^{(2)}_{\overline{48+\frac{48}{365}|}} = \frac{1}{2} \frac{1 - v^{96+\frac{96}{365}}_{(\cdot 01579)}}{\cdot 01579}.$$

To evaluate $v^{96+\frac{96}{365}}_{(\cdot 01579)}$, we have

$$\begin{aligned}
\log v^{96+\frac{96}{365}} &= \left(96 + \frac{96}{365}\right) \log v \\
&= -\left(96 + \frac{96}{365}\right) \log 1\cdot 01579 \\
&= -\left(96 + \frac{96}{365}\right) \times \cdot 006804 \\
&= \bar{1}\cdot 345027 \\
&= \log \cdot 22132
\end{aligned}$$

$$\text{Therefore} \quad a^{(2)}_{\overline{48+\frac{48}{365}|}} = \frac{1 - \cdot 22132}{\cdot 03158} = 24\cdot 657.$$

5. Each payment of a perpetuity is divisible equally among five funds. It is arranged that, instead of the perpetuity being shared as at present, four of the funds should for a fixed number of years each in succession, receive the annual payments in full, and that the fifth fund should be entitled to the perpetuity in full thereafter.

Find the number of years which must elapse before the fifth fund comes into possession.

Here the benefit which the fifth fund is procuring is a perpetuity deferred t years, and the benefit it is forgoing is a fifth part of a perpetuity, and these two must be equal.

Hence $$v^t \frac{1}{i} = \frac{1}{5}\frac{1}{i}$$

$$v^t = \cdot 2$$

and $$t = \frac{\log \cdot 2}{\log v}$$

6. Find the value at 3 per cent. of an annuity-certain for 30 years, the annual payment to be reduced by one-half after the end of each period of 10 years. Given v^{10} at 3 per cent. $= \cdot 74409$.

Here the value of the annuity may be written

$$a = a_{\overline{10}|} + \frac{1}{2} v^{10} a_{\overline{10}|} + \frac{1}{4} v^{20} a_{\overline{10}|}$$

Now $$a_{\overline{10}|} = \frac{1 - v^{10}}{i} = \frac{1 - \cdot 74409}{\cdot 03} = 8\cdot 530$$

$$v^{10} a_{\overline{10}|} = \cdot 74409 \times 8\cdot 530 = 6\cdot 347$$

and $$v^{20} a_{\overline{10}|} = \cdot 74409 \times 6\cdot 347 = 4\cdot 723$$

Therefore $$a = 8\cdot 530 + \frac{1}{2} \times 6\cdot 347 + \frac{1}{4} \times 4\cdot 723$$
$$= 12\cdot 884.$$

7. A shareholder in a life company holds £2000 of its paid-up capital, the dividends on which are increased 10 per cent. every quinquennial valuation. Supposing a valuation to have just taken place and the dividends for the next five years to be fixed at £100 per annum, what is the value of his interest in the undertaking upon a 5 per cent. basis?

The annual payment for the first five years is 100, for the next five $100 \times 1\cdot 1$, for the next five $100 \times (1\cdot 1)^2$, and so on. The present value of all these payments is

$$100 a_{\overline{5}|} + v^5 100 (1\cdot 1) a_{\overline{5}|} + v^{10} 100 (1\cdot 1)^2 a_{\overline{5}|} + \cdots$$
$$= 100 a_{\overline{5}|} \{ 1 + v^5 (1\cdot 1) + v^{10} (1\cdot 1)^2 + \cdots \}$$
$$= 100 a_{\overline{5}|} \frac{1}{1 - v^5 (1\cdot 1)}$$

which at 5 per cent. is equal to £3134·547.

8. Calculate the price to be charged for an annuity-certain of £25 for 30 years, assuming 4 per cent. interest for the first 10 years, decreasing thereafter by $\frac{1}{2}$ per cent. per annum each period of 10 years. (Use the Tables given at the end of the *Theory of Finance.*)

We have

$$\begin{aligned}
a &= a_{\overline{10}|\,(4\%)} + v^{10}_{(4\%)}\,a_{\overline{10}|\,(3\frac{1}{2}\%)} + v^{10}_{(4\%)}\,v^{10}_{(3\frac{1}{2}\%)}\,a_{\overline{10}|\,(3\%)} \\
&= 8{\cdot}11090 + ({\cdot}675564 \times 8{\cdot}31661) \\
&\qquad + ({\cdot}675564 \times {\cdot}708919 \times 8{\cdot}53020) \\
&= 8{\cdot}11090 + 5{\cdot}61840 + 4{\cdot}08528 \\
&= 17{\cdot}81458 \\
\text{and } 25 \times a &= 445{\cdot}3645 \\
&= \text{£}445,\ 7\text{s. } 3\text{d. nearly.}
\end{aligned}$$

9. In connection with a feu-duty of £20 per annum, a duplicand is payable every 21 years, the next being due 7 years hence. Find the present value at 4 per cent. of all future duplicands; and the equivalent addition to the feu-duty if all duplicands be dispensed with. Given $v^3 = {\cdot}88900$, $v^4 = {\cdot}85480$.

The present value of all future duplicands is

$$\begin{aligned}
&20\,(v^7 + v^{28} + v^{49} + \cdot\ \cdot\ \cdot\ ad\ inf.) \\
&= 20\,\frac{v^7}{1 - v^{21}}
\end{aligned}$$

Now $v^7 = v^3 \times v^4 = {\cdot}88900 \times {\cdot}85480 = {\cdot}75992$,

and $v^{21} = (v^7)^3 = {\cdot}43883$.

Therefore the value of the duplicands

$$\begin{aligned}
&= 20 \times \frac{{\cdot}75992}{1 - {\cdot}43883} \\
&= 27{\cdot}083.
\end{aligned}$$

This may be looked on as the benefit.

If, then, the annual addition to the feu-duty be P, we have the payment side

$$= \mathrm{P}\,\frac{1}{{\cdot}04} = 25\,\mathrm{P}.$$

Equating the two sides, we have

$$\begin{aligned}
25\,\mathrm{P} &= 27{\cdot}083, \\
\text{whence}\quad \mathrm{P} &= 1{\cdot}083. \\
&= \text{£}1,\ 1\text{s. } 8\text{d.}
\end{aligned}$$

10. The annual value of a certain property is £20, and this increases each year by 1 per cent. The property is subject to a feu-duty of £8 a year, but the feu-duty payable at the end of every 21st year from the present time is to be, not £8, but the full annual value of the property at that time. Give a formula for the present value of the feu-duty.

The value of the annual payment of £8 in perpetuity is obviously $\frac{8}{i}$.

But every 21st year the £8 is not receivable, and the deduction on this account is therefore

$$8\,(v^{21}+v^{42}+v^{63}+\;\cdot\;\cdot\;\cdot\;ad\ inf.)$$
$$=8\,\frac{v^{21}}{1-v^{21}}$$

Instead of the £8, there is receivable the full annual value of the property, at the end of every 21st year, and the present value of this is

$$20\,(1\cdot01)^{21}v^{21}+20\,(1\cdot01)^{42}v^{42}+20\,(1\cdot01)^{63}v^{63}+\;\cdot\;\cdot\;\cdot\;ad\ inf.$$
$$=\frac{20\,(1\cdot01)^{21}v^{21}}{1-(1\cdot01)^{21}v^{21}}$$

Therefore the full value of the feu-duty is expressed by the formula—

$$\frac{8}{i}-8\,\frac{v^{21}}{1-v^{21}}+20\frac{(1\cdot01)^{21}v^{21}}{1-(1\cdot01)^{21}v^{21}}$$

11. Construct a schedule showing the repayment of a loan of £1750 by means of an annuity-certain for 4 years payable half-yearly at 5 per cent. interest.

Half-Year.	Interest contained in each Payment.	Principal contained in each Payment.	Principal Repaid to Date.	Principal still Outstanding.
1	43·750	200·318	200·318	1549·682
2	38·742	205·326	405·644	1344·356
3	33·609	210·459	616·103	1133·897
4	28·348	215·720	831·823	918·177
5	22·954	221·114	1052·937	697·063
6	17·427	226·641	1279·578	470·422
7	11·761	232·307	1511·885	238·115
8	5·953	238·115	1750·000	...

12. Construct a similar schedule of the repayment of a loan of £1300 in 7 years at $3\frac{1}{2}$ per cent.

Year.	Interest in Annual Payment.	Principal in Annual Payment.	Principal Repaid to Date.	Principal still Outstanding.
1	45·500	167·108	167·108	1132·892
2	39·651	172·957	340·065	959·935
3	33·598	179·010	519·075	780·925
4	27·333	185·275	704·350	595·650
5	20·848	191·760	896·110	403·890
6	14·136	198·472	1094·582	205·418
7	7·190	205·418	1300·000	...

13. A loan of £10,000, bearing interest at the rate of 4 per cent. per annum, payable half-yearly, is to be repaid by 40 equal half-yearly payments, including interest and instalment of principal. Having given $(1{\cdot}02)^{-20} = {\cdot}67297$, find

(*a*) The amount of the half-yearly payment.

(*b*) The amount of principal included in the first and in the twenty-first half-yearly payments respectively.

(*c*) The total amount of principal repaid, after payment of the twentieth half-yearly sum.

(*a*) The half-yearly payment $= \dfrac{10000}{a_{\overline{40}|(2\%)}}$

$$= \frac{10000 \times {\cdot}02}{1 - v^{40}_{(2\%)}} = \frac{200}{1-({\cdot}67297)^2}$$

$$= 365{\cdot}557.$$

(*b*) The principal in the first payment

$$= \frac{10000}{a_{\overline{40}|}} \times v^{40} = 365{\cdot}557 \times ({\cdot}67297)^2$$

$$= 165{\cdot}557.$$

That in the twenty-first payment

$$= \frac{10000}{a_{\overline{40}|}} \times v^{20} = 365{\cdot}557 \times {\cdot}67297$$

$$= 246{\cdot}009.$$

(c) The total principal repaid after the twentieth payment

$$= 10000 - \frac{10000}{a_{\overline{40}|}} \times a_{\overline{20}|}$$

$$= 10000 - 365\cdot557 \times \frac{1 - \cdot67297}{\cdot02}$$

$$= 10000 - 5977\cdot405$$

$$= 4022\cdot595.$$

14. Given $a_{\overline{25}|}$ at 4 per cent. $= 15\cdot6221$, and $a_{\overline{10}|}$ at the same rate $= 8\cdot1109$, find the capital included in the 15th payment of the former annuity.

The capital included in the 15th payment of $a_{\overline{25}|}$

$$= v^{25-15+1} = v^{11}$$

the value of which is found as follows:—

$$a_{\overline{10}|} = \frac{1 - v^{10}}{i}$$

whence $v^{10} = 1 - ia_{\overline{10}|}$

$$= 1 - \cdot04 \times 8\cdot1109$$

$$= \cdot675564$$

Therefore $v^{11} = \frac{\cdot675564}{1\cdot04}$

$$= \cdot649581.$$

15. A life office advances £1000, repayable in 10 years, by an annuity to secure interest at the rate of 5 per cent., and provide for the accumulation of the sinking fund at the rate of 3 per cent. When the sixth annual payment becomes due, the borrower desires to cancel the arrangement and repay the loan at once. Find the amount of capital actually outstanding and state what sum you would advise the office to accept in satisfaction of its claim.

The capital outstanding is the original loan less the accumulation of sinking fund,

$$= 1000 - \frac{1000}{s_{\overline{10}|\,(3\%)}} \times (1\cdot03)s_{\overline{5}|\,(3\%)} = 522\cdot988$$

But the redemption money should be the value of an annuity-due,

of the same amount as the office was receiving, for 5 years at 3 per cent.

$$= 1000\left(\frac{1}{s_{\overline{10}|\,(3\%)}} + \cdot 05\right)\left(1 + a_{\overline{4}|\,(3\%)}\right) = 647\cdot 330$$

16. Given that the amount of an annuity-certain of 1 is 26·87037, and that the present value of the same annuity is 14·87748, find the rate of interest.

Using the formula

$$\frac{1}{a_{\overline{n}|}} - i = \frac{1}{s_{\overline{n}|}}$$

we have

$$i = \frac{1}{a_{\overline{n}|}} - \frac{1}{s_{\overline{n}|}}$$

In the present example

$$\begin{aligned} i &= \frac{1}{14\cdot 87748} - \frac{1}{26\cdot 87037} \\ &= \cdot 067216 - \cdot 037216 \\ &= \cdot 03. \end{aligned}$$

17. A perpetuity of £7, 10s. payable yearly, and a composition of £7, 10s. payable at the end of the 10th and every 20th year thereafter, are to be redeemed by an annuity payable half-yearly for 30 years. Find the amount of the annuity, taking interest at 4 per cent.

Here it will be convenient to find separate expressions for the value of the old benefit which is being given up and for the value of the consideration which is taking its place, equating the two thereafter to ascertain the amount of the annuity.

$$\begin{aligned} \text{The Benefit Side} &= 7\cdot 5\left\{\frac{1}{i} + (v^{10} + v^{30} + v^{50} + \cdots \textit{ad inf.})\right\} \\ &= 7\cdot 5\left(\frac{1}{i} + \frac{v^{10}}{1 - v^{20}}\right) \\ &= 7\cdot 5\left(25 + \frac{\cdot 675564}{\cdot 543613}\right) \\ &= 196\cdot 8205. \end{aligned}$$

Now, if P be the half-yearly payment, the Payment Side

$$\begin{aligned} &= \mathrm{P} \times a_{\overline{60}|\,(2\%)} \\ &= \mathrm{P} \times 34\cdot 7609 \end{aligned}$$

interest being assumed to be convertible at the periods of payment of the annuity.

Equating, we have

$$P \times 34{\cdot}7609 = 196{\cdot}8205$$

and
$$P = \frac{196{\cdot}8205}{34{\cdot}7609}$$
$$= 5{\cdot}662.$$
$$= £5,\ 13s.\ 3d.\ \text{nearly}.$$

18. If a sum of £1,000,000 be borrowed at 4 per cent. interest, payable annually, and £60,000 be applied each year towards paying the interest and reducing the principal, in what time will the loan be finally discharged?

Here we have $1,000,000 = 60,000\, a_{\overline{n|}}$ where n is unknown. To find n we proceed as follows:—

$$6a_{\overline{n|}} = 100$$
$$a_{\overline{n|}} = 16\tfrac{2}{3}$$
$$1 - v^n = \tfrac{2}{3}$$
$$v^n = \tfrac{1}{3}$$
$$n \log v = -\log 3$$
$$n = \frac{-\log 3}{-\log 1{\cdot}04}$$
$$= \frac{-{\cdot}4771213}{-{\cdot}0170333}$$
$$= 28{\cdot}01.$$

The time is therefore practically 28 years.

19. An annuity of £50, payable by half-yearly instalments for 20 years, is bought at 14 years' purchase. Find, approximately, the rate of interest realised by the purchaser.

It should be explained that, when "r years' purchase" is spoken of, it means that the price paid is r times the annual rent of the annuity. In this case £700.

[Effective rate = $3\tfrac{3}{4}$ per cent. almost.

20. An annuity-certain for 35 years is bought at 20 years' purchase. What rate of interest is made on the investment?

[Rate of interest = $3\tfrac{1}{2}$ per cent. very nearly.

21. An annuity of £80, payable in half-yearly instalments for 25 years, is bought for £1400. Required the half-yearly rate of interest which is made on the investment.

Here we have $1400 = 40\, a_{\overline{50}|}$

Hence $a_{\overline{50}|} = 35.$

Rate of interest $= 1\tfrac{1}{2}$ per cent. very nearly.

22. Find the rate of interest at which a is calculated when it $= 16{\cdot}938$, given annuity values for the same term as follows:—

3	per cent.	17·413
$3\tfrac{1}{2}$	per cent.	16·482
4	per cent.	15·622

Using formula (43) of this chapter in the *Theory of Finance*, we have

$$\rho = {\cdot}005 \times \frac{-{\cdot}931 + {\cdot}0355}{\dfrac{(-{\cdot}931)^2}{-{\cdot}475} + {\cdot}0355}$$

$$= {\cdot}005 \times \frac{{\cdot}8955}{1{\cdot}789\dot{3}}$$

$$= {\cdot}0025, \text{ approximately,}$$

whence $i = {\cdot}03 + \rho = {\cdot}0325$, or $3\tfrac{1}{4}$ per cent.

23. From the tables given at the end of the *Theory of Finance*, calculate the value at $3\tfrac{1}{4}$ per cent. of an annuity-certain for 20 years.

The formula to be used is

$$a_{(3\frac{1}{4}\%)} = a_{(3\%)} + \frac{\frac{1}{4}}{\frac{1}{2}}\,\Delta a_{(3\%)} + \frac{\frac{\frac{1}{4}}{\frac{1}{2}}\left(\frac{\frac{1}{4}}{\frac{1}{2}} - 1\right)}{\underline{|2}}\,\Delta^2 a_{(3\%)}$$

where $\Delta a_{(3\%)}$ and $\Delta^2 a_{(3\%)}$ represent the successive differences of $a_{(3\%)}$, $a_{(3\frac{1}{2}\%)}$, and $a_{(4\%)}$ for a period of 20 years.

The true value of $a_{\overline{20}|}$ at $3\tfrac{1}{4}$ per cent. is 14·539.

24. From the tables given at the end of the *Theory of Finance*, calculate the amount at $7\tfrac{1}{4}$ per cent. of an annuity-certain for 25 years payable half-yearly, interest convertible half-yearly.

The formula to follow is

$$s^{(2)}_{\overline{25}|\,(7\frac{1}{4}\%)} = \tfrac{1}{2}\, s_{\overline{50}|\,(3\frac{5}{8}\%)} = \tfrac{1}{2}\left\{ s_{(3\frac{1}{2}\%)} + \frac{\frac{1}{8}}{\frac{1}{2}}\,\Delta\, s_{(3\frac{1}{2}\%)} + \frac{\frac{\frac{1}{8}}{\frac{1}{2}}\left(\frac{\frac{1}{8}}{\frac{1}{2}}-1\right)}{\underline{|2}}\,\Delta^2 s_{(3\frac{1}{2}\%)}\right\}$$

where $\Delta\, s_{(3\frac{1}{2}\%)}$ and $\Delta^2 s_{(3\frac{1}{2}\%)}$ represent the successive differences of $s_{(3\frac{1}{2}\%)}$, $s_{(4\%)}$, and $s_{(4\frac{1}{2}\%)}$ for a period of 50 years.

The true value of $\frac{1}{2}\, s_{\overline{50}|}$ at $3\frac{5}{8}$ per cent. is 68·032.

Neither in this question nor in the previous one will the formula quite give the true value of the function, as second differences are assumed to be constant.

25. Assuming one rate of interest throughout, obtain prospectively and retrospectively the value of a Capital-Redemption Policy of 1 taken out n years ago for a period of t years at an annual premium of $P_{\overline{t}|}$, and prove the identity of the two expressions.

Before attempting this question, the student should know something of prospective and retrospective policy-values, though he will come more in contact with them when discussing life-policies at a later period.

When a capital-redemption policy is entered upon, the value of the benefit to be ultimately received is exactly equal to the value of the series of premiums to be paid therefor. As time goes on and the date of payment approaches, the value of the capital sum obviously increases, while on the other hand the premiums to be paid are fewer and their value consequently decreases. Thus the value of the benefit now exceeds the value of the premiums still to be paid. For this difference the office must keep a sum in hand which is called the "policy-value." The policy-value has here been looked at from the "prospective" point of view.

But again, after the policy has been in force for a number of years, the premiums which the office has received have been invested and accumulated (at the rate of interest assumed in the calculations). These accumulations constitute the value of the policy, which has here been discussed from a "retrospective" point of view.

In the case in the above question the value of the benefit at

the commencement of the contract is v^t; and the value of the premiums is $P_{\overline{t|}}(1+a_{\overline{t-1|}})$, whence $P_{\overline{t|}} = \dfrac{v^t}{1+a_{\overline{t-1|}}}$

After n years the value of the benefit is increased to v^{t-n}, while the value of the premiums is reduced to $P_{\overline{t|}}(1+a_{\overline{t-n-1|}})$. Therefore prospectively for the value of the policy we have

$$_nV_{\overline{t|}} = v^{t-n} - P_{\overline{t|}}(1+a_{\overline{t-n-1|}})$$

Again, the n premiums already paid have accumulated to $P_{\overline{t|}}(1+i)s_{\overline{n|}}$, and therefore retrospectively

$$_nV_{\overline{t|}} = P_{\overline{t|}}(1+i)s_{\overline{n|}}$$

These two expressions are identical, for

$$\begin{aligned} v^{t-n} - P_{\overline{t|}}(1+a_{\overline{t-n-1|}}) &= v^{t-n} - \frac{v^t}{1+a_{\overline{t-1|}}}(1+a_{\overline{t-n-1|}}) \\ &= v^t\left[\frac{(1+i)^n\{(1+i)-v^{t-1}\}-\{(1+i)-v^{t-n-1}\}}{(1+a_{\overline{t-1|}})\times i}\right] \\ &= \frac{v^t}{1+a_{\overline{t-1|}}} \times \frac{(1+i)\{(1+i)^n-1\}}{i} \\ &= P_{\overline{t|}}(1+i)s_{\overline{n|}} \end{aligned}$$

26. Calculate the net level annual premium for a capital-redemption assurance of £100 payable at the expiration of 50 years, assuming $3\frac{1}{2}$ per cent. interest for the first 10 years, 3 per cent. for the next 20 years, and $2\frac{1}{2}$ per cent. thereafter.

The Benefit Side $= 100 \times v^{10}_{(3\frac{1}{2}\%)} \times v^{20}_{(3\%)} \times v^{20}_{(2\frac{1}{2}\%)}$

The Payment Side

$$= P\left(a_{\overline{10|}(3\frac{1}{2}\%)} + v^{10}_{(3\frac{1}{2}\%)} \times a_{\overline{20|}(3\%)} + v^{10}_{(3\frac{1}{2}\%)} \times v^{20}_{(3\%)} \times a_{\overline{20|}(2\frac{1}{2}\%)}\right)$$

Therefore

$$\begin{aligned} P &= \frac{100 \times v^{10}_{(3\frac{1}{2}\%)} \times v^{20}_{(3\%)} \times v^{20}_{(2\frac{1}{2}\%)}}{a_{\overline{10|}(3\frac{1}{2}\%)} + \left(v^{10}_{(3\frac{1}{2}\%)} \times a_{\overline{20|}(3\%)}\right) + \left(v^{10}_{(3\frac{1}{2}\%)} \times v^{20}_{(3\%)} \times a_{\overline{20|}(2\frac{1}{2}\%)}\right)} \\ &= \frac{100 \times \cdot 7089188 \times \cdot 5536758 \times \cdot 6102710}{8 \cdot 6076866 + \cdot 7089188(15 \cdot 3237994 + \cdot 5536758 \times 15 \cdot 9788911)} \\ &= \frac{23 \cdot 954}{25 \cdot 743} \\ &= \cdot 931 \\ &= 18s.\ 8d.\ \text{nearly.} \end{aligned}$$

27. Find the annual premium per cent. for a Leasehold Assurance to mature at the end of 30 years,

(*a*) assuming 3 per cent. interest throughout;

(*b*) assuming 3 per cent. for the first 20 years and $2\frac{1}{2}$ per cent. thereafter.

Answers: (*a*) £2, 0s. 10d. per cent.
(*b*) £2, 2s. 8d. per cent.

28. It is desired to have a policy providing £1000 at the end of 30 years. The policy is to be by annual premiums under a special system which provides for the premium being doubled at the end of 5 years. Calculate at 3 per cent. interest the premium payable during the first 5 years.

Here we may state the Benefit Side as $1000\,v^{30} = 411{\cdot}987$, and the Payment Side as

$$\mathrm{P}\{(1+a_{\overline{29}|})+v^5(1+a_{\overline{24}|})\} \text{ or } \mathrm{P}\{2(1+a_{\overline{29}|})-(1+a_{\overline{4}|})\}$$
$$= \mathrm{P}\times 35{\cdot}660.$$

Equating the Benefit Side to the Payment Side, we get

$$\mathrm{P}\times 35{\cdot}660 = 411{\cdot}987,$$

whence $\mathrm{P} = 11{\cdot}553$
$= £11,\ 11s.\ 1d.$ nearly.

29. Express in the simplest form for applying to Interest Tables the annual premium required to provide £1000 at the end of $3n$ years, the premium to be reduced by one-half from the beginning of each n years.

The simplest formula for this premium is

$$\mathrm{P} = \frac{1000\,v^{3n}}{a_{\overline{3n}|}+a_{\overline{2n}|}+2a_{\overline{n}|}}$$

Then the premium for the first n years is 4P, for the second n years 2P, and for the remainder of the period P.

30. Calculate the reserve required at the end of 30 years under a Leasehold Assurance policy with a premium of £10—(*a*) assuming $2\frac{1}{2}$ per cent. throughout; (*b*) assuming $3\frac{1}{4}$ per cent. for 10 years, decreasing thereafter by $\frac{1}{4}$ per cent. per annum each period of 10 years.

(*a*) The reserve required is (by the retrospective method)

$$10(s_{\overline{31}|(2\frac{1}{2}\%)} - 1) = 450.$$

(*b*) The reserve here is

$$10\{(s_{\overline{11}|(3\frac{1}{2}\%)} - 1)(1\cdot03)^{10}(1\cdot0275)^{10} + (s_{\overline{11}|(3\%)} - 1)(1\cdot0275)^{10} + (s_{\overline{11}|(2\frac{3}{4}\%)} - 1)\} = 482.$$

31. An insurance office calculates its Leasehold Assurance premiums at 3 per cent. interest, and allows as the surrender value of a yearly-premium policy the premiums paid, with the exception of the first, accumulated at 3 per cent. interest, less a deduction of 10 per cent. Given v^{20} at 3 per cent. $= \cdot55368$, find—

(*a*) The annual premium required to provide a Leasehold Assurance policy for £100 payable at the end of 20 years;

(*b*) The surrender value allowed by the office for such a policy at the end of 10 years.

$$(a) \qquad P_{\overline{20}|} = \frac{v^{20}}{(1+i)a_{\overline{20}|}} = \frac{\cdot55368 \times \cdot03}{(1\cdot03)(1-\cdot55368)}$$

$$= £3,\ 12s.\ 3d.\ \text{per cent. nearly.}$$

$$(b) \qquad \text{S.V.} = \cdot9\{3\cdot613 \times (s_{\overline{10}|} - 1)\}$$

$$\text{Now } s_{\overline{10}|} = \frac{a_{\overline{10}|}}{v^{10}} = \frac{1-v^{10}}{iv^{10}}$$

$$\text{and } v^{10} = (v^{20})^{\frac{1}{2}} = (\cdot55368)^{\frac{1}{2}}$$

$$= \cdot74410.$$

$$\text{Therefore } s_{\overline{10}|} = \frac{1-\cdot74410}{\cdot03 \times \cdot74410}$$

$$= 11\cdot464,$$

$$\text{and S.V.} = \cdot9 \times 3\cdot613 \times 10\cdot464$$

$$= 34\cdot026$$

$$= £34,\ 0s.\ 6d.\ \text{nearly.}$$

32. If A represents the fund of a life assurance company at the beginning of the year, B the fund at the end of the year, and I the amount received for interest during the year, find the rate of interest realised by the office during the year.

$\dfrac{I}{\frac{A+B}{2}}$ represents the instantaneous rate or force of interest,

being the annual rate per unit at which the funds are increasing by interest at any moment of time, assuming that the increase or decrease in the funds is uniform throughout the year.

$\dfrac{I}{\frac{A+B}{2} - \frac{1}{2}I}$ represents the effective rate of interest or the rate of interest actually realised by the company during the year.

33. An estate, the clear annual value of which is £800, is let by a college at a rent of £300 per annum on a lease for 20 years, which may be renewed at the end of 7 years on payment of a sum of money. Interest being reckoned at 6 per cent., what sum should the tenant pay on renewing his lease? Given log 106 = 2·0253059, log 4·688385 = ·6710233, and log 3·118042 = ·4938820.

The lease has 13 years to run; the tenant wishes the term extended to 20 years. Therefore, if he is to continue at the same annual rent, he must pay the difference between the full annual value and the rent for the period of extension, or

$$
\begin{aligned}
500_{13|}a_{\overline{7}|} &= 500(a_{\overline{20}|} - a_{\overline{13}|}) \\
&= 500\left(\frac{1-v^{20}}{i} - \frac{1-v^{13}}{i}\right) \\
&= 500\left(\frac{v^{13}-v^{20}}{i}\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{Now}\quad \log(1{\cdot}06)^{-13} &= -13\log 1{\cdot}06 \\
&= -13 \times {\cdot}0253059 \\
&= -{\cdot}3289767 \\
&= \bar{1}{\cdot}6710233 \\
&= \log {\cdot}4688385 \\
\text{And}\quad \log(1{\cdot}06)^{-20} &= -20 \times {\cdot}0253059 \\
&= -{\cdot}5061180 \\
&= \bar{1}{\cdot}4938820 \\
&= \log {\cdot}3118042
\end{aligned}
$$

$$
\begin{aligned}
\text{Therefore}\quad 500_{13|}a_{\overline{7}|} &= 500 \times \frac{{\cdot}4688385 - {\cdot}3118042}{{\cdot}06} \\
&= 1308{\cdot}619.
\end{aligned}
$$

The sum to be paid by the tenant is thus £1308, 12s. 5d. nearly.

CHAPTER III

Varying Annuities

1. In the scheme of figurate numbers it is to be noted that the mth term of any order is equal to the sum of the first $(m-1)$ terms of the preceding order.

Again, from the consideration that the terms of the $(r-1)$th ordeı are the first differences of the terms of the rth, those of the $(r-2)$th are the second differences, and so on, and those of the first are the $(r-1)$th differences of the terms of the rth order, and from the formula

$$u_m = u_1 + (m-1)\Delta u_1 + \frac{(m-1)(m-2)}{\underline{|2}}\Delta^2 u_1 + \cdots$$
$$+ \frac{(m-1)(m-2)\cdots(m-r+1)}{\underline{|r-1}}\Delta^{r-1}u_1 + \cdots$$

we have

$$t_{\overline{m|}\,\overline{r|}} = t_{\overline{1|}\,\overline{r|}} + (m-1)\Delta t_{\overline{1|}\,\overline{r|}} + \frac{(m-1)(m-2)}{\underline{|2}}\Delta^2 t_{\overline{1|}\,\overline{r|}} + \cdots$$
$$+ \frac{(m-1)(m-2)\cdots(m-r+1)}{\underline{|r-1}}\Delta^{r-1} t_{\overline{1|}\,\overline{r|}}$$

(all higher differences vanishing)

$$= t_{\overline{1|}\,\overline{r|}} + (m-1)t_{\overline{1|}\,\overline{r-1|}} + \frac{(m-1)(m-2)}{\underline{|2}}t_{\overline{1|}\,\overline{r-2|}} + \cdots$$
$$+ \frac{(m-1)(m-2)\cdots(m-r+1)}{\underline{|r-1}}t_{\overline{1|}\,\overline{1|}}$$

But the first terms of all orders except the first are zero, and the first term of the first order is 1. Therefore

$$t_{\overline{m|}\,\overline{r|}} = \frac{(m-1)(m-2)\cdots(m-r+1)}{\underline{|r-1}}$$

2. A proof by induction of the value of $a_{\overline{n}|\,\overline{r}|}$ is as follows:—

We must premise that

$$a_{\overline{n}|\,\overline{r}|} = v\,a_{\overline{n-1}|\,\overline{r-1}|} + v^2 a_{\overline{n-2}|\,\overline{r-1}|} + \cdots + v^{n-1} a_{\overline{1}|\,\overline{r-1}|}$$

which may easily be proved by expressing $a_{\overline{n}|\,\overline{r}|}$ in simplest terms of v and i, and rearranging so as to obtain the right-hand side of the above equation.

Then we have

$$a_{\overline{n}|\,\overline{1}|} = \frac{1-v^n}{i}$$

$$\begin{aligned}
a_{\overline{n}|\,\overline{2}|} &= v\,a_{\overline{n-1}|\,\overline{1}|} + v^2 a_{\overline{n-2}|\,\overline{1}|} + \cdots + v^{n-1} a_{\overline{1}|\,\overline{1}|} \\
&= v\,\frac{1-v^{n-1}}{i} + v^2\frac{1-v^{n-2}}{i} + \cdots + v^{n-1}\,\frac{1-v}{i} \\
&= \frac{a_{\overline{n-1}|\,\overline{1}|} - (n-1)v^n}{i} \\
&= \frac{a_{\overline{n}|\,\overline{1}|} - n\,v^n}{i}
\end{aligned}$$

$$\begin{aligned}
a_{\overline{n}|\,\overline{3}|} &= v\,a_{\overline{n-1}|\,\overline{2}|} + v^2 a_{\overline{n-2}|\,\overline{2}|} + \cdots + v^{n-1} a_{\overline{1}|\,\overline{2}|} \\
&= v\frac{a_{\overline{n-1}|\,\overline{1}|} - (n-1)v^{n-1}}{i} + v^2\,\frac{a_{\overline{n-2}|\,\overline{1}|} - (n-2)v^{n-2}}{i} + \\
&\qquad \cdots + v^{n-1}\,\frac{a_{\overline{1}|\,\overline{1}|} - v}{i} \\
&= \frac{a_{\overline{n}|\,\overline{2}|} - \dfrac{n(n-1)}{2}\,v^n}{i}
\end{aligned}$$

Here the law of the formula seems to be disclosed, and we may assume that

$$a_{\overline{n}|\,\overline{r}|} = \frac{a_{\overline{n}|\,\overline{r-1}|} - \dfrac{n(n-1)\cdots(n-r+2)}{\underline{|r-1}}\,v^n}{i}$$

To obtain the value of $a_{\overline{n}|\,\overline{r+1}|}$ we must add to the right-hand side the difference between $a_{\overline{n}|\,\overline{r}|}$ and $a_{\overline{n}|\,\overline{r+1}|}$, or otherwise

$$a_{\overline{n}|\,\overline{r+1}|} = \frac{a_{\overline{n}|\,\overline{r-1}|} - v^n t_{\overline{n+1}|\,\overline{r}|}}{i} + (a_{\overline{n}|\,\overline{r+1}|} - a_{\overline{n}|\,\overline{r}|})$$

$$= \frac{a_{\overline{n}|\,\overline{r-1}|} - v^n t_{\overline{n+1}|\,\overline{r}|} + (1+i)(a_{\overline{n}|\,\overline{r+1}|} - a_{\overline{n}|\,\overline{r}|}) - (a_{\overline{n}|\,\overline{r+1}|} - a_{\overline{n}|\,\overline{r}|})}{i}$$

$$= \frac{1}{i}\Big\{(a_{\overline{n-1}|\,\overline{r-1}|} + v^n t_{\overline{n}|\,\overline{r-1}|}) - v^n t_{\overline{n+1}|\,\overline{r}|} + (\overline{1+i}\, a_{\overline{n+1}|\,\overline{r+1}|}$$

$$- v^n t_{\overline{n+1}|\,\overline{r+1}|}) - (a_{\overline{n-1}|\,\overline{r-1}|} + v\, a_{\overline{n-2}|\,\overline{r-1}|} + v^2 a_{\overline{n-3}|\,\overline{r-1}|} + \cdots + v^{n-2} a_{\overline{1}|\,\overline{r-1}|})$$

$$- (v\, a_{\overline{n-1}|\,\overline{r}|} + v^2 a_{\overline{n-2}|\,\overline{r}|} + \cdots + v^{n-1} a_{\overline{1}|\,\overline{r}|}) + (a_{\overline{n-1}|\,\overline{r}|} + v^n t_{\overline{n}|\,\overline{r}|})\Big\}$$

$$= \frac{1}{i}\Big\{v^n t_{\overline{n}|\,\overline{r-1}|} + v^n t_{\overline{n}|\,\overline{r}|} - v^n t_{\overline{n+1}|\,\overline{r}|} - v^n t_{\overline{n+1}|\,\overline{r+1}|}$$

$$+ (a_{\overline{n}|\,\overline{r}|} + v\, a_{\overline{n-1}|\,\overline{r}|} + v^2 a_{\overline{n-2}|\,\overline{r}|} + \cdots + v^{n-1} a_{\overline{1}|\,\overline{r}|})$$

$$- (v\, a_{\overline{n-1}|\,\overline{r}|} + v^2 a_{\overline{n-2}|\,\overline{r}|} + \cdots + v^{n-1} a_{\overline{1}|\,\overline{r}|}) - a_{\overline{n-1}|\,\overline{r}|} + a_{\overline{n-1}|\,\overline{r}|}\Big\}$$

$$= \frac{1}{i}(a_{\overline{n}|\,\overline{r}|} - v^n t_{\overline{n+1}|\,\overline{r+1}|})$$

$$= \frac{a_{\overline{n}|\,\overline{r}|} - \dfrac{n(n-1)\cdots(n-r+1)}{\underline{|r}}\, v^n}{i}$$

which follows the same law as the expression we assumed for $a_{\overline{n}|\,\overline{r}|}$, and which has been obtained therefrom by assuming nothing but the truth of the equation above premised. Therefore if the expression holds for the rth order it also holds for the $(r+1)$th. But we have seen that it holds for the third order, therefore it holds for the fourth; therefore for the fifth; and so on, until we reach the rth order, when we have the general expression

$$a_{\overline{n}|\,\overline{r}|} = \frac{a_{\overline{n}|\,\overline{r-1}|} - \dfrac{n(n-1)\cdots(n-r+2)}{\underline{|r-1}}\, v^n}{i}$$

3. The following may serve as an alternative explanation of the value of an annuity of the rth order as found by general reasoning—

Suppose one is entitled to a perpetuity of the $(r-1)$th order, $a_{\overline{\infty}|\,\overline{r-1}|}$, but prefers not to spend the payments as they fall due. Instead, they are invested, and the interest on the investments

alone is spent. Now for the first $(r-2)$ years nothing is received and nothing can be invested. At the end of the $(r-1)$th year a payment of $t_{\overline{r-1|}\,\overline{r-1|}}$ is received, which is invested, and yields in a year interest of $i\,t_{\overline{r-1|}\,\overline{r-1|}}$, or $i\,t_{\overline{r|}\,\overline{r|}}$. This amount is spent at the end of the rth year.

But, further, at the end of the rth year a payment of $t_{\overline{r|}\,\overline{r-1|}}$ is received, which is invested along with the previous $t_{\overline{r-1|}\,\overline{r-1|}}$, and the interest received at the end of the $(r+1)$th year is

$$i\left(t_{\overline{r-1|}\,\overline{r-1|}}+t_{\overline{r|}\,\overline{r-1|}}\right), \text{ or } i\,t_{\overline{r+1|}\,\overline{r|}},$$

since the mth term of any order is equal to the sum of the first $(m-1)$ terms of the preceding order. This amount of $i\,t_{\overline{r+1|}\,\overline{r|}}$ is spent at the end of the $(r+1)$th year.

The payment of $t_{\overline{r+1|}\,\overline{r-1|}}$ receivable at this time is also invested and the whole interest received at the end of the $(r+2)$th year, $i(t_{\overline{r+1|}\,\overline{r|}} + t_{\overline{r+1|}\,\overline{r-1|}})$ or $i\,t_{\overline{r+2|}\,\overline{r|}}$, is spent.

This process goes on in perpetuity. But we notice that the annuity being spent is $i\,a_{\overline{\infty|}\,\overline{r|}}$

We therefore have

$$a_{\overline{\infty|}\,\overline{r-1|}} = i a_{\overline{\infty|}\,\overline{r|}}$$

and

$$a_{\overline{\infty|}\,\overline{r|}} = \frac{a_{\overline{\infty|}\,\overline{r-1|}}}{i}$$

If, however, the payments receivable cease at the end of n years, we have an annuity for n years, and in this case we must take account of the payments of the annuity of the $(r-1)$th order which have been held back and not spent. The sum of these at the end of n years is

$$\left(t_{\overline{r-1|}\,\overline{r-1|}} + t_{\overline{r|}\,\overline{r-1|}} + t_{\overline{r+1|}\,\overline{r-1|}} + \cdots + t_{\overline{n|}\,\overline{r-1|}}\right)$$

which is of course equal to $t_{\overline{n+1|}\,\overline{r|}}$

We thus see that the value of an annuity for n years of the $(r-1)$th order is equal to the value of i times an annuity for n years of the rth order plus the value of a payment of $t_{\overline{n+1|}\,\overline{r|}}$ due n years hence. In symbols

$$a_{\overline{n|}\,\overline{r-1|}} = i\,a_{\overline{n|}\,\overline{r|}} + v^n t_{\overline{n+1|}\,\overline{r|}}$$

whence

$$a_{\overline{n|}\,\overline{r|}} = \frac{a_{\overline{n|}\,\overline{r-1|}} - v^n t_{\overline{n+1|}\,\overline{r|}}}{i}$$

4. As explained in Article 25, Finite Differences are of great use in finding the value of any varying annuity, and it is here also that the use of the values of annuities according to the several orders of figurate numbers comes in. For it will be observed that, in stating u_1, u_2, u_3, etc., in terms of u_1 and its successive differences, the coefficients of the differences follow precisely the same laws as the scheme of orders shown in *Theory of Finance*, Article 3.

Thus we have

$$v\,u_1 + v^2u_2 + v^3u_3 + \cdot\cdot\cdot + v^n u_n$$
$$= a_{\overline{n}|\overline{1}|}\,u_1 + a_{\overline{n}|\overline{2}|}\,\Delta u_1 + a_{\overline{n}|\overline{3}|}\,\Delta^2 u_1 + \cdot\cdot + a_{\overline{n}|\overline{r}|}\,\Delta^{r-1}u_1$$

where the rth and higher differences of the series u_1, u_2, u_3, etc., vanish.

If the series be a perpetuity, we have, as is shown,

$$v\,u_1 + v^2u_2 + v^3u_3 + \cdot\cdot\cdot \; ad\ inf.$$
$$= a_{\overline{\infty}|\overline{1}|}\,u_1 + a_{\overline{\infty}|\overline{2}|}\,\Delta u_1 + a_{\overline{\infty}|\overline{3}|}\,\Delta^2 u_1 + \cdot\cdot\cdot + a_{\overline{\infty}|\overline{r}|}\,\Delta^{r-1}u_1$$

where the rth and higher differences vanish

$$= \frac{u_1}{i} + \frac{\Delta u_1}{i^2} + \frac{\Delta^2 u_1}{i^3} + \cdot\cdot\cdot + \frac{\Delta^{r-1}u_1}{i^r}$$

From this we may get a formula for the value of the series of payments u_1, u_2, u_3, $\cdot\cdot\cdot\, u_n$, as follows :—

$$v\,u_1 + v^2u_2 + v^3u_3 + \cdot\cdot\cdot + v^n u_n$$
$$= \left(\frac{u_1}{i} + \frac{\Delta u_1}{i^2} + \cdot\cdot\cdot + \frac{\Delta^{r-1}u_1}{i^r}\right)$$
$$- v^n\left(\frac{u_{n+1}}{i} + \frac{\Delta u_{n+1}}{i^2} + \cdot\cdot\cdot + \frac{\Delta^{r-1}u_{n+1}}{i^r}\right)$$

This will be found useful where the differences are not numerous, and the number of terms unknown.

As an example, suppose it is required to find the value of the annuity whose payments are 1, 5, 11, $\cdot\cdot\cdot$ 109.

Here $u_1 = 1$, $\Delta u_1 = 4$, $\Delta^2 u_1 = 2$, and

$$109 = u_n = u_1 + (n-1)\Delta u_1 + \frac{(n-1)(n-2)}{2}\Delta^2 u_1$$
$$= 1 + 4(n-1) + (n-1)(n-2)$$
$$= n^2 + n - 1$$

whence $n = 10$.

Also $u_{11}=131$, $u_{12}=155$, $\Delta u_{11}=24$, $\Delta^2 u_{11}=\Delta^2 u_1=2$.
Therefore

$$v+5v^2+11v^3+\cdots+109v^{10}$$
$$=\left(\frac{1}{i}+\frac{4}{i^2}+\frac{2}{i^3}\right)-v^{10}\left(\frac{131}{i}+\frac{24}{i^2}+\frac{2}{i^3}\right).$$

EXAMPLES

1. A 20-year sinking-fund policy is effected on an increasing scale of premiums, beginning at £100 per annum and rising by £3 each year till the end of the term. At the end of the fourth year it is proposed to commute further payments. Determine their value on a 3 per cent. basis.

The premium due at the beginning of the 5th year is £112, which will increase by £3 per annum for each of the succeeding 15 years. Now, assuming for the moment that the premiums will be payable at the end and not the beginning of each year, we have their value equal to

$$112\,a_{\overline{16}|\,\overline{1}|}+3\,a_{\overline{16}|\,\overline{2}|}$$

Adjusting this expression (since the premiums are actually payable at the beginning of each year) by multiplying it by $(1+i)$, we have the commutation price of the future premiums equal to

$$(1+i)(112\,a_{\overline{16}|\,\overline{1}|}+3\,a_{\overline{16}|\,\overline{2}|})$$
$$=(1+i)\left(112a_{\overline{16}|}+3\,\frac{a_{\overline{16}|}-16v^{16}}{i}\right)$$
$$=1{\cdot}03\left(112\times12{\cdot}56110+3\,\frac{12{\cdot}56110-16\times{\cdot}623167}{{\cdot}03}\right)$$
$$=1{\cdot}03\left(1406{\cdot}8432+3\,\frac{12{\cdot}56110-9{\cdot}97067}{{\cdot}03}\right)$$
$$=1{\cdot}03(1406{\cdot}8432+259{\cdot}043)$$
$$=£1715{\cdot}863.$$

2. Prove that if $a_{\overline{n}|\,\overline{r}|}$ denote the value of a varying annuity of the rth order for n years then $(1+i)^r\,a_{\overline{n}|\,\overline{r}|}$ is equal to the sum of the first $(n-r+1)$ terms of the expansion of $(1-v)^{-r}$ in powers of v.

Remembering that

$$t_{\overline{m}|\overline{r}|} = \frac{(m-1)(m-2)\cdot\cdot\cdot(m-r+1)}{\underline{|r-1}} = {}_{m-1}C_{r-1}$$

and that where $m < r$, $t_{\overline{m}|\overline{r}|} = 0$, we have $(1+i)^r a_{\overline{n}|\overline{r}|}$

$$= (1+i)^r(v\,t_{\overline{1}|\overline{r}|} + v^2 t_{\overline{2}|\overline{r}|} + v^3 t_{\overline{3}|\overline{r}|} + \cdot\cdot\cdot + v^n t_{\overline{n}|\overline{r}|})$$

$$= (1+i)^r({}_{r-1}C_{r-1}\,v^r + {}_rC_{r-1}\,v^{r+1} + \cdot\cdot\cdot + {}_{n-1}C_{r-1}\,v^n)$$

$$= (1+i)^r\left\{v^r + rv^{r+1} + \frac{(r+1)r}{\underline{|2}}v^{r+2} + \cdot\cdot\cdot + \frac{(n-1)(n-2)\cdot\cdot\,2\cdot1}{\underline{|n-r}\,\underline{|r-1}}v^n\right\}$$

$$= 1 + rv + \frac{r(r+1)}{\underline{|2}}v^2 + \cdot\cdot\cdot + \frac{r(r+1)\cdot\cdot(n-1)}{\underline{|n-r}}v^{n-r}$$

which is the sum of the first $(n-r+1)$ terms of the expansion of $(1-v)^{-r}$ in powers of v.

3. Find at 4 per cent. the value of the annuity of which the first three payments are 40, 45, and 52 respectively, and the last 325.

The first step necessary is to find the term of the annuity. For this, we use the formula

$$u_n = u_1 + (n-1)\Delta u_1 + \frac{(n-1)(n-2)}{2}\Delta^2 u_1 + \cdot\cdot\cdot$$

In this case

$325 = 40 + 5(n-1) + (n-1)(n-2)$, since $\Delta u_1 = 5$ and $\Delta^2 u_1 = 2$

Hence
$$n^2 + 2n - 288 = 0$$
$$(n+18)(n-16) = 0$$
$$n = 16.$$

Now, proceeding to find the value of the annuity, we have

$$\begin{aligned} a &= 40\,a_{\overline{16}|} + 5\,a_{\overline{16}|\overline{2}|} + 2\,a_{\overline{16}|\overline{3}|} \\ &= (40 \times 11{\cdot}652) + (5 \times 77{\cdot}744) + (2 \times 341{\cdot}879) \\ &= 1538{\cdot}558. \end{aligned}$$

4. What is the value at 5 per cent. of the annuity whose payments are 16, 26, 58, 124, etc., the sum of all the payments being 1322480?

Here it is necessary to find n, the term, from the formula

$$\Sigma u_n = n\,u_1 + \frac{n(n-1)}{\underline{|2}}\Delta u_1 + \frac{n(n-1)(n-2)}{\underline{|3}}\Delta^2 u_1 + \cdot\cdot\cdot$$

Then proceed to apply the formula of varying annuities—

$$a = u_1 a_{\overline{n}|\overline{1}|} + \Delta u_1 a_{\overline{n}|\overline{2}|} + \Delta^2 u_1 a_{\overline{n}|\overline{3}|} + \Delta^3 u_1 a_{\overline{n}|\overline{4}|}$$

The value required is 287998·936, as follows:—

$$\begin{aligned} a &= 16\, a_{\overline{40}|\overline{1}|} + 10\, a_{\overline{40}|\overline{2}|} + 22\, a_{\overline{40}|\overline{3}|} + 12\, a_{\overline{40}|\overline{4}|} \\ &= (16 \times 17{\cdot}159) + (10 \times 229{\cdot}545) \\ &\qquad + (22 \times 2374{\cdot}991) + (12 \times 19431{\cdot}595) \\ &= 274{\cdot}544 + 2295{\cdot}450 + 52249{\cdot}802 + 233179{\cdot}140 \\ &= 287998{\cdot}936. \end{aligned}$$

5. Find the present value of an annuity of the rth order, to yield interest at the rate i per annum on the whole capital for the entire term of the annuity, the capital to be replaced by means of a sinking fund accumulating at the rate j per annum.

Here we must resort to the general rule given in Article 43*a* of Chapter II. that the present value of any series of payments, n remaining constant, may be found "by multiplying the amount, accumulated at rate j to the end of the n years, of the series of payments by $\dfrac{1}{1+is'_{\overline{n}|}}$."

Now the amount of an annuity of the rth order accumulated at rate j is $s'_{\overline{n}|\overline{r}|}$ and we therefore have the value of an annuity of the rth order under the conditions laid down as $\dfrac{s'_{\overline{n}|\overline{r}|}}{1+is'_{\overline{n}|}}$ where $s'_{\overline{n}|\overline{r}|}$ and $s'_{\overline{n}|}$ are taken at rate j.

6. Find the value of an annuity-certain for 20 years whose several payments are 1, 2, 3 · · · 20, the value to be so calculated as to yield the purchaser 5 per cent. on his whole investment throughout the whole of the 20 years and to return him his capital at 3 per cent.

Applying the rule cited in the preceding example, we have

$$a = \frac{s_{\overline{20}|\overline{1}|\,(3\%)} + s_{\overline{20}|\overline{2}|\,(3\%)}}{1 + {\cdot}05\, s_{\overline{20}|\,(3\%)}}$$

since the amount of the annuity to the end of 20 years is $(s_{\overline{20}|\,\overline{1}|}+s_{\overline{20}|\,\overline{2}|})$ at 3 per cent.

$$= \frac{26{\cdot}87037+229{\cdot}01248}{1+1{\cdot}34352}$$

$$= 109{\cdot}187.$$

7. A loan of £42,000 has been made with the following condition as to repayment: Annual instalments of principal for 20 years, the first being £4000, the second £3800, the third £3600, and so on; interest at the rate of 4 per cent. being paid annually on the outstanding amounts. Immediately after payment of the fifth instalment, it is arranged to repay the balance of the advance with a premium, the lenders being able to re-invest at only 3 per cent. Show how the premium should be computed.

The value of the capital at 3 per cent. is

$$3000a_{\overline{15}|\,\overline{1}|(3\%)}-200a_{\overline{15}|\,\overline{2}|(3\%)}$$

and of the interest on the outstanding amounts of capital

$$\cdot 04(24000a_{\overline{15}|\,\overline{1}|(3\%)}-3000a_{\overline{15}|\,\overline{2}|(3\%)}+200a_{\overline{15}|\,\overline{3}|(3\%)}).$$

Therefore the whole value of the outstanding loan is

$$3960a_{\overline{15}|\,\overline{1}|(3\%)}-320a_{\overline{15}|\,\overline{2}|(3\%)}+8a_{\overline{15}|\,\overline{3}|(3\%)}.$$

And the premium required is

$$3960a_{\overline{15}|\,\overline{1}|(3\%)}-320a_{\overline{15}|\,\overline{2}|(3\%)}+8a_{\overline{15}|\,\overline{3}|(3\%)}-24000.$$

CHAPTER IV

Loans Repayable by Instalments

1. In connection with the discussion of loans repayable by instalments, it is important to remember that the symbol C stands for the capital actually returnable by the borrower, that is, taking account of any discount or premium on the par value of the loan, and further, that j is dependent on this definition of C, being the ratio which the annual payment of interest bears to C. These two points must be clearly borne in mind in all questions of this nature.

2. The following wording, differing slightly from that of Mr King, may be useful to explain the general formula for the value of a loan, repayable by instalments at stated periods of time, with interest in the meantime at rate j, so as to yield the purchaser a given rate of interest, i.

Had the borrower contracted to pay interest at rate i per annum on the capital C, then the required value of the loan would have necessarily been C, since, interest at rate i being payable at the end of each year, with C repayable, the investor would have realised rate i on the purchase price. The value of the capital at rate i being K, the value at rate i of the interest, on the basis assumed, would have been $(C-K)$, which then is the value of the annual payments of interest, if these were made at rate i. But, in point of fact, the annual payments of interest are made at rate j, and therefore by simple proportion their value is $\frac{j}{i}(C-K)$. Adding to the present value of the interest the present value of the capital as already noted, namely K, we have the whole value of the loan equal to $K+\frac{j}{i}(C-K)$.

3. In the converse problem to find the rate of interest yielded by a loan purchased at a given price, we have the equation $i=\frac{j(C-K)}{A-K}$, whence i may be found as explained in Articles 11 and 12.

Mr Ralph Todhunter has given (*J. I. A.* xxxiii. 356) a very useful formula for approximating to the rate of interest yielded by a bond bought at a premium. In the explanation accompanying the formula, he suggests that the premium might be dealt with in practice by writing down the book-value out of each dividend by an equal proportionate part of the premium, the remainder of the dividend being treated as interest. Taking the case of a bond, repayable at par at the end of n years, interest meantime at rate j, purchased at a premium of p per unit, Mr Todhunter gives us the following schedule :—

Year.	Book-Value of Bond at beginning of Year.	Interest.
1	$1+p$	$j-\frac{p}{n}$
2	$1+\frac{n-1}{n}p$	$j-\frac{p}{n}$
3	$1+\frac{n-2}{n}p$	$j-\frac{p}{n}$
$\vdots$	$\vdots$	$\vdots$
$n-1$	$1+\frac{2}{n}p$	$j-\frac{p}{n}$
n	$1+\frac{1}{n}p$	$j-\frac{p}{n}$

giving an average rate of interest

$$i=\frac{j-\frac{p}{n}}{1+\frac{1}{n}\left(p+\frac{n-1}{n}p+\frac{n-2}{n}p+\cdots+\frac{2}{n}p+\frac{1}{n}p\right)}$$

$$=\frac{j-\frac{p}{n}}{1+\frac{1}{n}p\frac{n(n+1)}{2n}}$$

$$=\frac{j-\frac{p}{n}}{1+\frac{n+1}{2n}p}$$

The result by this formula will as a rule be sufficiently close for practical purposes; but if greater accuracy be desired, the rate of interest thus found may be used to obtain K in the formula $i = \frac{j(C-K)}{A-K}$, when a very close approximation would be obtained.

Mr Todhunter, however, points out that his formula should only be applied as a final result when n and j are not large. The reason for this will be readily understood from the following:—

Using Makeham's formula, we have

$$i = \frac{j(C-K)}{A-K}$$

or in the example submitted by Mr Todhunter

$$i = \frac{j(1-v^n)}{1+p-v^n}$$

where v^n is calculated at rate i.

Hence $\quad i(1-v^n)+pi = j(1-v^n)$

and

$$\begin{aligned}
pi &= (j-i)(1-v^n) \\
j-i &= pi\{1-(1+i)^{-n}\}^{-1} \\
&= pi\left\{ni - \frac{n(n+1)}{\lfloor 2}i^2 + \frac{n(n+1)(n+2)}{\lfloor 3}i^3 - \dots\right\}^{-1} \\
&= \frac{p}{n}\left\{1 - \frac{n+1}{\lfloor 2}i + \frac{(n+1)(n+2)}{\lfloor 3}i^2 - \dots\right\}^{-1} \\
&= \frac{p}{n}\left(1 + \frac{n+1}{2}i + \frac{n^2-1}{12}i^2 - \dots\right)
\end{aligned}$$

Neglecting higher powers of i than the first, we have

$$j-i = \frac{p}{n}\left(1 + \frac{n+1}{2}i\right)$$

whence
$$i = \frac{j - \frac{p}{n}}{1 + \frac{n+1}{2n}p}$$

which is Mr Todhunter's formula as given above.

EXAMPLES

1. A bond for £775, 15s. 10d., repayable at par in five years, and bearing interest at $3\frac{3}{4}$ per cent. payable half-yearly (first payment six months hence), is bought at a price (£750) to yield the investor $4\frac{1}{2}$ per cent. on his investment. Draw up a schedule showing the amounts that must be added to capital each half-year so as to gradually write up the sum invested to the redemption value.

The schedule will be as follows :—

Half-Year. (1.)	Book-Value at beginning of Half-Year (2.)	Interest on Book-Value at $2\frac{1}{4}$%. (3.)	Interest received at $1\frac{7}{8}$% on £775, 15s 10d (4.)	Amount to be added to Book-Value. (3)−(4) (5.)	Book-Value at end of Half-Year. (2)+(5) (6.)	Half-Year. (7.)
1	750·000	16·875	14·546	2·329	752·329	1
2	752·329	16·927	14·546	2·381	754·710	2
3	754·710	16·981	14·546	2·435	757·145	3
4	757·145	17·035	14·546	2·489	759·634	4
5	759·634	17·092	14·546	2·546	762·180	5
6	762·180	17·149	14·546	2·603	764·783	6
7	764·783	17·207	14·546	2·661	767·444	7
8	767·444	17·267	14·546	2·721	770·165	8
9	770·165	17·328	14·546	2·782	772·947	9
10	772·947	17·391	14·546	2·845	775·792	10

2. The value to yield 4 per cent. of a 5 per cent. bond for £1000 due after five years is £1044, 10s. 4d. Give a schedule of the amounts to be carried each year to principal and interest, with the amount of principal outstanding at the beginning of each year.

Here the bond's book-value has to be written down (not up, as in the former case), and the schedule will accordingly be :—

Year. (1.)	Book-Value at beginning of Year. (2.)	Interest on Book-Value at 4%. (3.)	Interest received at 5% on £1000. (4.)	Amount carried to Principal. (4)−(3) (5.)	Book-Value at end of Year. (2)−(5) (6.)	Year. (7.)
1	1044·517	41·781	50·000	8·219	1036·298	1
2	1036·298	41·452	50·000	8·548	1027·750	2
3	1027·750	41·110	50·000	8·890	1018·860	3
4	1018·860	40·755	50·000	9·245	1009·615	4
5	1009·615	40·385	50·000	9·615	1000·000	5

3. The five per cent. stock of a colonial municipality is redeemable at par in 20 years. What can a purchaser give for it in order to make 4 per cent. on his investment?

Using Makeham's formula,

$$A = K + \frac{j}{i}(C - K)$$

we have $K = 100v^{20}_{(4\%)}$, $j = \cdot 05$, $i = \cdot 04$, and $C = 100$.

Therefore

$$A = 100\left\{v^{20}_{(4\%)} + \frac{\cdot 05}{\cdot 04}\left(1 - v^{20}_{(4\%)}\right)\right\} = 100\left(\cdot 456387 + \frac{5}{4} \times \cdot 543613\right)$$

$$= 113\cdot 590.$$

4. Two loans were granted 10 years ago—(*a*) £20,000 at $4\frac{3}{4}$ per cent. per annum nominal, repayable by 60 equal half-yearly instalments which include both principal and interest; (*b*) £20,000 at $4\frac{1}{2}$ per cent. per annum nominal, repayable by 60 equal half-yearly payments of principal, interest being also paid on the balance from time to time remaining outstanding. Find in each case the amount of the payment due to-day, and show the amounts of principal and interest included in the payment. Find also the sum for which each loan may be redeemed to-day, assuming interest at $3\frac{1}{2}$ per cent. per annum nominal.

First, as to the payments due to-day—

(*a*) In this case the periodical payment is always the same, and is equal to $\dfrac{20000}{a_{\overline{60}|\,(2\frac{3}{8}\%)}}$ whereof $\dfrac{20000}{a_{\overline{60}|}} \times v^{41}_{(2\frac{3}{8}\%)}$ is principal, and the remainder, or $\dfrac{20000}{a_{\overline{60}|}}\left(1 - v^{41}_{(2\frac{3}{8}\%)}\right)$, is interest.

(*b*) The payment due now consists of (1) $\dfrac{20000}{60}$ of principal and (2) interest on the balance of principal outstanding at the beginning of the twentieth half-year, that is

$$\left(20000 - 19\,\frac{20000}{60}\right)\cdot 0225.$$

Thus the whole payment due is $20000\left\{\dfrac{1}{60} + \left(1 - \dfrac{19}{60}\right)\cdot 0225\right\}$

Secondly, as to the redemption price—

(*a*) As the amount paid to redeem the loan can only be invested at 3½ per cent., the future half-yearly payments must be valued at that rate, that is, the price is

$$\frac{20000}{a_{\overline{60}|(2\frac{1}{4}\%)}} \times (1 + a_{\overline{40}|(1\frac{3}{4}\%)})$$

(*b*) The value of the future instalments of capital is $20000 \times \frac{1}{60} a_{\overline{40}|(1\frac{3}{4}\%)}$, and of the future payments of interest $20000\left(\frac{40}{60} \times \cdot 0225\, a_{\overline{40}|\,\overline{1}|(1\frac{3}{4}\%)} - \frac{1}{60} \times \cdot 0225\, a_{\overline{40}|\,\overline{2}|(1\frac{3}{4}\%)}\right)$. Therefore the redemption price is

$$20000\left\{\left(\frac{1}{60} + \frac{40}{60} \times \cdot 0225\right) a_{\overline{40}|\,\overline{1}|(1\frac{3}{4}\%)} - \frac{1}{60} \times \cdot 0225\, a_{\overline{40}|\,\overline{2}|(1\frac{3}{4}\%)}\right\}$$

plus the payment due to-day.

5. A 6 per cent. debenture of £100, redeemable at par in 20 years, is sold for £107, 10s. How would you approximate to the rate of interest realised by the purchaser, given that to obtain 5 per cent. he would have paid £112, 9s. 3d.?

Since we are given the value of A to return 5 per cent. to the purchaser, we can find the value of K from the equation

$$112 \cdot 463 = K + \frac{\cdot 06}{\cdot 05}(100 - K)$$

$$\begin{aligned}
\text{For} \qquad K(\tfrac{6}{5} - 1) &= 120 - 112 \cdot 463 \\
\text{and} \qquad K &= 37 \cdot 685 \\
\text{But} \qquad K &= 100\, v^{20}_{(5\%)} \\
\text{therefore} \qquad v^{20}_{(5\%)} &= \cdot 37685.
\end{aligned}$$

Now the question is to find the rate of interest when the purchase price is £107, 10s.

We have $\qquad i = j\,\dfrac{C - K}{A - K}$

or substituting 37·685 for K (that is, approximating with 5 per cent.)

$$\begin{aligned}
i &= \cdot 06\,\frac{62 \cdot 315}{69 \cdot 815} \\
&= \cdot 0536
\end{aligned}$$

from which we see that 5 per cent. is too low, the true rate being about 5⅜ per cent.

6. A bond of £100, bearing interest at the rate of 4 per cent. per annum, payable half-yearly, and redeemable at the expiration of 30 years at a premium of 10 per cent., is bought for £114. Find approximately the rate of interest realised by the investor, having given $a_{\overline{60}|}$ at $1\frac{3}{4}$ per cent. = 36·964.

In the formula $i = j\dfrac{C-K}{A-K}$, i and j for this question denote half-yearly interest, $j = \cdot 02 \times \dfrac{100}{110} = \dfrac{\cdot 02}{1\cdot 1}$, $C = 110$, $K = 110v^{60}$, and $A = 114$.

$$\text{Since} \qquad \frac{1-v^{60}}{\cdot 0175} = 36\cdot 964$$

$$v^{60} = \cdot 35313$$

Therefore trying $1\frac{3}{4}$ per cent., we have

$$i = \frac{\cdot 02}{1\cdot 1} \times \frac{110 - 38\cdot 8443}{114 - 38\cdot 8443}$$

$$= \frac{\cdot 02}{1\cdot 1} \times \cdot 94678$$

$$= \cdot 01721.$$

Hence the approximate yearly rate of interest is

$$(1\cdot 01721)^2 - 1 = \cdot 03472, \text{ or } 3\cdot 472 \text{ per cent.}$$

7. A bond for £1000, bearing interest at 5 per cent. payable half-yearly, and repayable at par in 30 years, is purchased for £1250. What rate of interest does the investment yield to the purchaser?

As before, we have

$$i = j\frac{C-K}{A-K}$$

where it will be well to treat i and j as interest for a half-year. We then have

$$i = \cdot 025\,\frac{1000 - 1000\,v^{60}}{1250 - 1000\,v^{60}}$$

Trying v^{60} first at 2 per cent. we get

$$i = \cdot 025\,\frac{1000 - 304\cdot 782}{1250 - 304\cdot 782}$$

$$= \cdot 0184$$

Seeing that 2 per cent. is too large, we try again with $1\frac{3}{4}$ per cent. and get

$$i = \cdot 025\,\frac{1000 - 353\cdot 130}{1250 - 353\cdot 130}$$
$$= \cdot 0180.$$

Now a reduction of $\frac{1}{4}$ per cent. makes a change of $\cdot 04$ per cent. in the result. Therefore to get the true rate we have

$$2 - x = 1\cdot 84 - \frac{\cdot 04}{\cdot 25}x$$

whence $x = \cdot 19$

and $i = \cdot 0181.$

This being the half-yearly rate, the effective yearly rate is

$$(1\cdot 0181)^2 - 1 = \cdot 0365, \text{ or } 3\cdot 65 \text{ per cent.}$$

8. Towards the close of 1905 the Japanese Government issued a four per cent. loan at 90 per cent., repayable at par on 1st January 1931 (or in certain circumstances earlier), coupons for a half-year's interest payable 1st January and 1st July each year, with first payment on 1st July 1906. Allowing for discount on the instalments the issue-price may be taken as $89\frac{1}{2}$ on 1st January 1906. Find the rate of interest realised.

$$i = j\,\frac{C - K}{A - K}$$
$$= \cdot 02\,\frac{100 - 100\,v^{50}}{89\cdot 5 - 100\,v^{50}}$$
$$= (\text{taking } v^{50} \text{ at } 2\tfrac{1}{2} \text{ per cent.} = \cdot 29094)\ \cdot 0235$$
$$= (\text{taking } v^{50} \text{ at } 2\tfrac{1}{4} \text{ per cent.} = \cdot 32873)\ \cdot 0237$$

whence approximately the half-yearly rate is $\cdot 0236$ and the effective rate is $(1\cdot 0236)^2 - 1 = \cdot 04776$ = about £4, 15s. 6d. per cent.

9. A debenture of £100, redeemable at £110 on 1st July 1915, and bearing interest at the rate of $4\frac{1}{2}$ per cent. per annum, payable half-yearly on 1st January and 1st July in each year, is purchased on 1st April 1905 for £109. How would you calculate the yield to the purchaser?

Here i must be considered as the rate yielded to the purchaser per half-year. At 1st July 1905 the value of the capital will be

$110 \times v^{20}_{(i)}$, and of the interest $2{\cdot}25(1 + a_{\overline{20}|(i)})$. As that date is half a period forward we must discount these values for that time, and we get the equation

$$109\,(1+i)^{\frac{1}{2}} = 110v^{20}_{(i)} + 2{\cdot}25(1 + a_{\overline{20}|(i)}).$$

i must now be approximated to, and this result being the half-yearly rate we obtain the effective yearly rate from $(1+i)^2 - 1$.

It is probably better, however, to proceed in the same way as in our other examples. Then, i being the half-yearly rate as before, we have

$$i = \frac{{\cdot}0225}{1{\cdot}1} \times \frac{110(1+i)^{\frac{1}{2}} - 110\,v^{20\frac{1}{2}}}{109 - 110\,v^{20\frac{1}{2}}}$$

As before, i may be approximated to, and the yearly rate found from the result.

The formulas produce the same result.

10. A bond for £100, bearing interest at 6 per cent. per annum payable half-yearly, and redeemable at par at the end of 40 years from the date of issue, was issued at par 30 years ago, the present market value being £115.

(*a*) Find the rate of interest yielded to a purchaser now buying at the market price.

(*b*) Find the rate of interest obtained by the original holder when his profit on sale is taken into account.

(*c*) If it were proposed to convert the bond into one for £125 bearing $3\frac{1}{2}$ per cent. interest, redeemable at par in 30 years, what gain or loss would there be to the holder of the bond on conversion?

$$(a) \quad i = j \times \frac{C-K}{A-K} = {\cdot}03 \times \frac{100 - 100\,v^{20}}{115 - 100\,v^{20}}$$

Trying 2 per cent.,

$$\begin{aligned} i &= {\cdot}03 \times \frac{100 - 67{\cdot}297}{115 - 67{\cdot}297} \\ &= \frac{{\cdot}03 \times 32{\cdot}703}{47{\cdot}703} \\ &= 2{\cdot}0567 \text{ per cent.} \end{aligned}$$

Also trying $2\frac{1}{4}$ per cent.,

$$
\begin{aligned}
i &= \cdot 03 \times \frac{100 - 64\cdot 082}{115 - 64\cdot 082} \\
&= \frac{\cdot 03 \times 35\cdot 918}{50\cdot 918} \\
&= 2\cdot 1162 \text{ per cent.}
\end{aligned}
$$

$$
\begin{aligned}
2\cdot 25 &\text{ gives } 2\cdot 1162 \\
\text{and } \quad 2 &\text{ gives } 2\cdot 0567.
\end{aligned}
$$

Thus a difference of $\cdot 25$ gives a difference of $\cdot 0595$.

$$
\begin{aligned}
\text{Therefore} \quad 2 + x &= 2\cdot 0567 + \frac{\cdot 0595}{\cdot 25}x \\
\text{whence} \quad x &= \cdot 0744 \\
\text{and} \quad i &= 2\cdot 0744 \text{ per cent.}
\end{aligned}
$$

The yearly rate then $= (1\cdot 020744)^2 - 1 = \cdot 04192 = 4\cdot 192$ per cent.

$$
(b) \quad i = \frac{\cdot 03}{1\cdot 15} \times \frac{115 - 115\,v^{60}}{100 - 115\,v^{60}}
$$

Trying 3 per cent.,

$$
\begin{aligned}
i &= \frac{\cdot 03}{1\cdot 15} \times \frac{115 - 115 \times \cdot 16973}{100 - 115 \times \cdot 16973} \\
&= \frac{\cdot 03(100 - 16\cdot 973)}{100 - 19\cdot 519} \\
&= 3\cdot 0949 \text{ per cent.}
\end{aligned}
$$

Also trying $3\frac{1}{4}$ per cent.,

$$
\begin{aligned}
i &= \frac{\cdot 03(100 - 14\cdot 676)}{100 - 115 \times \cdot 14676} \\
&= \frac{\cdot 03 \times 85\cdot 324}{83\cdot 123} \\
&= 3\cdot 0794 \text{ per cent.}
\end{aligned}
$$

A difference of $\cdot 25$ in the rate per cent. gives a difference of $-\cdot 0155$.

$$
\begin{aligned}
\text{Therefore} \quad 3 + x &= 3\cdot 0949 - \frac{\cdot 0155}{\cdot 25}x \\
\text{whence} \quad x &= \cdot 0894 \\
\text{and} \quad i &= 3\cdot 0894 \text{ per cent.}
\end{aligned}
$$

and the yearly rate $= (1\cdot 030894)^2 - 1 = \cdot 06274 = 6\cdot 274$ per cent.

$$
\begin{aligned}
(c) \quad A &= K + \frac{j}{i}(C - K) \\
&= 125\,v^{30} + \frac{\cdot 035}{i}(125 - 125\,v^{30})
\end{aligned}
$$

Where $i = \cdot 04192$, as found in (a)

$$A = 36\cdot465 + \frac{\cdot035}{\cdot04192}(125 - 36\cdot465)$$
$$= 36\cdot465 + 73\cdot920$$
$$= 110\cdot385.$$

Thus there is a loss of 4·615, or, say, £4, 12s. 4d. to the holder of the bond on conversion on the terms given.

11. A foreign corporation issues a loan of £390,000 4 per cent. bonds, repayable by annual drawings as follows:—£10,000 at the end of 5 years, £11,000 at the end of 6 years, £12,000 at the end of 7 years, and so on, till the whole is repaid. The issue-price being 94½ per cent., what rate of interest is paid by the corporation?

Here it is necessary to find the term when the last instalment is paid. We have

$$390 = 10 + 11 + 12 + \cdot\;\cdot\;\cdot + \{10 + (n-1)\}$$
$$= \frac{20 + (n-1)}{2}n$$

whence $n = 20$.

Therefore the value of the instalments of capital is

$$1000v^4(10\,a_{\overline{20}|\,\overline{1}|} + a_{\overline{20}|\,\overline{2}|}).$$

Using now the formula $i = j\dfrac{C-K}{A-K}$ and trying 4½ per cent., we have $K = 196943\cdot6$, as found below.

Also

$$C = 390000$$
$$A = 390000 \times \cdot945 = 368550$$

and $j = \cdot04$.

Therefore

$$i = \cdot04 \times \frac{390000 - 196943\cdot6}{368550 - 196943\cdot6}$$
$$= \cdot045.$$

$$\begin{array}{rcr}
a_{\overline{20}|\,\overline{1}|} & = & 13\cdot00794 \\
20\,v^{20} & = & \underline{8\cdot29286} \\
\div\cdot045 & & \underline{4\cdot71508} \\
a_{\overline{20}|\,\overline{2}|} & = & 104\cdot7796 \\
10\,a_{\overline{20}|\,\overline{1}|} & = & \underline{130\cdot0794} \\
 & & 234\cdot8590 \\
\times\,1000v^4 & & \underline{838\cdot561} \\
K & = & \underline{\underline{196943\cdot6}}
\end{array}$$

Thus the rate is 4½ per cent.

12. Having given the value of $a_{\overline{20|}}$ at $4\frac{1}{2}$ per cent. = 13·0079 and at 5 per cent. = 12·4622, find approximately the rate of interest yielded by an annuity for 20 years, in which the payments are successively 20, 19, 18, etc., when purchased for £150.

The successive terms of this annuity involve first differences only, and consequently its value may be stated in the symbols

$$20\,a_{\overline{20|}\,\overline{1|}} - a_{\overline{20|}\,\overline{2|}}$$

Now at $4\frac{1}{2}$ per cent. $a_{\overline{20|}\,\overline{1|}} = 13·0079$,

$$v^{20} = 1 - i\,a_{\overline{20|}} = 1 - ·585355 = ·414645$$

$$\text{and}\quad a_{\overline{20|}\,\overline{2|}} = \frac{a_{\overline{20|}\,\overline{1|}} - 20\,v^{20}}{·045} = \frac{13·0079 - 8·2929}{·045} = 104·778$$

Therefore $\quad 20\,a_{\overline{20|}\,\overline{1|}} - a_{\overline{20|}\,\overline{2|}} = 260·158 - 104·778 = 155·380$

Again, at 5 per cent. $a_{\overline{20|}\,\overline{1|}} = 12·4622$.

$$v^{20} = 1 - i\,a_{\overline{20|}} = 1 - ·623110 = ·37689$$

$$\text{and}\quad a_{\overline{20|}\,\overline{2|}} = \frac{a_{\overline{20|}\,\overline{1|}} - 20\,v^{20}}{·05} = \frac{12·4622 - 7·5378}{·05} = 98·488$$

Therefore $\quad 20\,a_{\overline{20|}\,\overline{1|}} - a_{\overline{20|}\,\overline{2|}} = 249·244 - 98·488 = 150·756$

We see that a rise of ·005 in the rate means a fall of 4·624 in the price, and to bring the price from 155·380 to 150 (a fall of 5·380) the rate of interest must be increased by

$$\frac{·005 \times 5·380}{4·624} = ·00582$$

Therefore approximately at the price of 150 the rate of interest realised is 5·082 per cent.

13. Apply Todhunter's formula to determine the rate of interest yielded by a terminable 6 per cent. debenture, repayable at par at the end of 20 years, purchased for £119, 10s.

Here we use the formula—

$$i = \frac{j - \frac{p}{n}}{1 + \frac{n+1}{2n}p}$$

$$= \frac{\cdot 06 - \frac{\cdot 195}{20}}{1 + \frac{21}{40} \times \cdot 195}$$

$$= \cdot 04558$$

$$= £4,\ 11\text{s. } 2\text{d. per cent.}$$

14. Twenty years ago a Local Board borrowed £100,000 at 5 per cent. from an Assurance Company, such loan being repayable by 30 equal annual payments, including principal and interest. The Board now offers $3\frac{3}{4}$ per cent. debentures, repayable at par 60 years hence, but now issued at 90, in equitable fulfilment of the contract with the Assurance Company. What amount in debentures should the Company accept?

The annual payment made under the contract of the original loan is $\frac{100000}{a_{\overline{30}|}(5\%)}$, and there are 10 such payments still to be made. If the Company is to forgo the receipt of these they should be commuted at the rate of interest presently ruling in the market, and the proceeds should be employed in buying the new debentures at 90.

To find the rate of interest now obtainable on investments we may take the rate yielded by these debentures as fair. Thus we have

$$i = j\frac{C - K}{A - K}$$

where $j = \cdot 0375$, $C = 100$, $A = 90$, and $K = 100\,v^{60}$ at, say, $4\frac{1}{4}$ per cent. $= 8\cdot 231$

$$= \frac{\cdot 0375(100 - 8\cdot 231)}{90 - 8\cdot 231}$$

$$= \cdot 0421 \text{ approximately.}$$

The value of the 10 annual payments outstanding is therefore $\frac{100000}{a_{\overline{30}|(5\%)}} \times a_{\overline{10}|(4\cdot21\%)}$, and the amount of debentures this sum will purchase at the price of 90 is

$$\frac{\frac{100000}{a_{\overline{30}|(5\%)}} \times a_{\overline{10}|(4\cdot21\%)}}{\cdot9} = \frac{100000 \times 8\cdot02677}{15\cdot37245 \times \cdot9}$$

$$= 58016\cdot994 = £58,016, \ 19s. \ 11d. \text{ nearly.}$$

15. A Company has an issue of 6 per cent. debentures maturing after 5 years, which are quoted at a price which yields 4 per cent., and it proposes to redeem them by issuing 5 per cent. debentures for the same nominal amount in lieu. Show how to find the number of years for which the 5 per cent. debentures should run so that the holders would still realise 4 per cent. on their investment.

In the general formula

$$A = K + \frac{j}{i}(C - K)$$

when $C = 1$, we have

$$A = 1 - (1 - K)\left(1 - \frac{j}{i}\right)$$

Under the present arrangement of 6 per cent. debentures

$$A = 1 - (1 - v^5_{(4\%)})\left(1 - \frac{\cdot06}{\cdot04}\right)$$

$$= 1 + \cdot02\, a_{\overline{5}|(4\%)}$$

On the proposed altered basis of 5 per cent. debentures

$$A = 1 - (1 - v^n_{(4\%)})\left(1 - \frac{\cdot05}{\cdot04}\right)$$

$$= 1 + \cdot01\, a_{\overline{n}|(4\%)}$$

Equating these two

$$1 + \cdot01\, a_{\overline{n}|(4\%)} = 1 + \cdot02\, a_{\overline{5}|(4\%)}$$

$$\cdot01(1 - v^n_{(4\%)}) = \cdot02(1 - v^5_{(4\%)})$$

Therefore $$v^n_{(4\%)} = 2\,v^5_{(4\%)} - 1$$

and $$n = \frac{\log(2\,v^5_{(4\%)} - 1)}{\log v}$$

$$= \frac{\bar{1}\cdot 80879}{\bar{1}\cdot 98297}$$

$$= 11\cdot 23.$$

The period may therefore be put at nearly $11\frac{1}{4}$ years.

CHAPTER V

Interest Tables

1. In Articles 16 and 17 is discussed the formation of a table of $\log(1+i)^n$. The column of values thus formed may be used further to get the values of $(1+i)^n$ by taking the natural numbers corresponding to the logs. As this is not done by a Continued Process, a periodical check is not sufficient. Each value must be separately checked. This may best be done by taking independently the logs of the table last found and comparing the results with the original logs.

2. If a column of $(1+i)^n$ has been formed, a column of v^n may be obtained from it by the use of a table of reciprocals. By taking the reciprocals of the values thus found the original table of $(1+i)^n$ should be reproduced, and a check put upon the work.

Again, a column of $\log v^n$ might be formed in the same way as a table of $\log(1+i)^n$ (in this case there is no need to start at the end of the table as for forming a table of v^n directly) and the natural numbers corresponding would be the values of v^n, each of which requires to be checked as before.

3. In view of what has been said about Leasehold Assurances it will be well to discuss methods of forming a table of annual premiums for such. It is necessary to note that as the premiums are due at the beginning of the year, we have to deal with annuities-due throughout.

First, we have $\quad P_{\overline{n|}} = \dfrac{v^n}{\mathrm{a}_{\overline{n|}}} = \dfrac{v^n}{1+a_{\overline{n-1|}}}$

and hence $\quad \log P_{\overline{n|}} = \log v^n - \log(1+a_{\overline{n-1|}}).$

Therefore, being supplied with values of v^n and $a_{\overline{n|}}$, we proceed to form columns of $\log v^n$ and $\log(1+a_{\overline{n-1|}})$ and, deducting the

value in the latter of these from the value in the former, we obtain $\log P_{\overline{n}|}$, from which $P_{\overline{n}|}$ may be obtained at once.

The following schedule shows the process :—

Term n. (1)	$\log v^n$ (2)	$\log(1+a_{\overline{n-1}|})$ (3)	$\log P_{\overline{n}|} =$ (2)−(3) (4)	$P_{\overline{n}|} =$ $\log^{-1}(4)$ (5)
1				
2				
3				
4				
etc.				

Again, we have

$$P_{\overline{n}|} = \frac{v^n}{a_{\overline{n}|}} = \frac{1}{(1+i)^n a_{\overline{n}|}} = \frac{1}{(1+i)s_{\overline{n}|}} = \frac{1}{s_{\overline{n+1}|} - 1}$$

Therefore knowing the values of $s_{\overline{n}|}$, we form therefrom a column of $s_{\overline{n+1}|} - 1$ and take the reciprocals of these, the results being the values of $P_{\overline{n}|}$. The following schedule indicates the method :—

Term n. (1)	$s_{\overline{n+1}|} - 1$ (2)	$P_{\overline{n}|} = \frac{1}{(2)}$ (3)
1		
2		
3		
4		
etc.		

Further, $$P_{\overline{n}|} = \frac{v^n}{a_{\overline{n}|}} = \frac{1-(1-v^n)}{1+a_{\overline{n-1}|}} = \frac{1}{1+a_{\overline{n-1}|}} - \frac{i\,a_{\overline{n}|}}{(1+i)a_{\overline{n}|}}$$

$$= \frac{1}{1+a_{\overline{n-1}|}} - d$$

It is on this relationship that Orchard's Conversion Tables for annual premiums are founded. By them if one enters with the given value of $a_{\overline{n-1|}}$ the result is $P_{\overline{n|}}$. They are fully discussed in Chapter VIII. of the *Text Book,* Part II., and are only mentioned here to show their use in forming a table of $P_{\overline{n|}}$.

In all the above methods it has been assumed that one rate of interest holds throughout the term of the assurance, but it is not implied that this assumption always holds good.

INSTITUTE OF ACTUARIES' TEXT BOOK—PART II.

CHAPTER I

The Mortality Table

1. A Mortality Table is defined in this Chapter as an instrument by means of which are measured the probabilities of life and the probabilities of death. In its final form a mortality table sets forth the history of the experience by means of the number living and the number dying columns. If we refer to page 494 of the *Text Book*, we find the following figures:—

Age.	Number Living.	Number Dying.
0	127,283	14,358
1	112,925	3,962
2	108,963	2,375
3	106,588	1,646
4	104,942	1,325
etc.	etc.	etc.

These figures tell us that, according to this experience, out of every 127,283 persons born, 112,925 on the average survive to age one, 108,963 on the average survive to age two, and so on. Or again, they tell us that out of the same number of births, 14,358 on the average die before attaining age one, 3,962 on the average attain age one, but die before attaining age two, and so on.

It cannot be too carefully impressed upon the student that a

mortality table does not give absolute but only relative or average results; in other words, it is not intended to be inferred from these figures that 127,283 children were in reality found who were all born at the same moment of time, that 14,358 died before attaining age one, that 3,962 actually attained age one but died before attaining age two, and so on. An arbitrary figure called the radix is selected to represent the number of entrants at the initial age, and the figures submitted are only on the average, and relative to one another.

Were we asked to form a mortality table representing the experience of Edinburgh during the calendar year 1906, it would not be sufficient to give us merely the deaths that occurred in Edinburgh during that calendar year, arranged according to year of age. The summation of these deaths would have no relation whatever to the l_0 persons out of which the d_0 deaths actually occurred, nor again would the $(l_0 - d_0)$ persons have any relation to the l_1 persons out of which the d_1 deaths occurred, and so on.

It is possible that the deaths column so supplied us might adopt a quite irregular form, for it naturally depends on the number living at each age out of which the deaths occurred. For example, the deaths between twenty and twenty-one might be twice as numerous as those between twenty-one and twenty-two, owing to the fact that the number living between twenty and twenty-one happened to be fully twice as numerous as those living between twenty-one and twenty-two.

Again, it would not be sufficient that it be added to our data that the number born in each calendar year for many years past had been equal to the annual deaths. The migration element would require to be kept before us, since people might be emigrating and immigrating in different numbers and at entirely different ages.

Before a mortality table can be formed in the way here discussed, it is essential that the population be proved to be in every way stationary; that is, that the annual births be equal to the deaths, that the births all take place on the same day of the year, say 1st January, and that there be no emigration or immigration.

EXAMPLES

1. Prove that $m_x = q_x + \frac{(q_x)^2}{2} + \frac{(q_x)^3}{4} + \ldots$

We have by *Text Book* formula (9)

$$q_x = \frac{2m_x}{2+m_x}$$

$$\text{whence } m_x = \frac{2q_x}{2-q_x}$$

$$= \frac{q_x}{1-\frac{1}{2}q_x}$$

$$= q_x(1-\tfrac{1}{2}q_x)^{-1}$$

$$= q_x + \frac{(q_x)^2}{2} + \frac{(q_x)^3}{4} + \ldots$$

2. Prove that $p_x = 1 - m_x + \frac{1}{2}(m_x)^2 - \frac{1}{4}(m_x)^3 + \ldots$
and that $q_x = m_x - \frac{1}{2}(m_x)^2 + \frac{1}{4}(m_x)^3 - \frac{1}{8}(m_x)^4 + \ldots$

From *Text Book* formula (8), we have

$$p_x = \frac{1-\frac{1}{2}m_x}{1+\frac{1}{2}m_x}$$

$$= (1-\tfrac{1}{2}m_x)(1+\tfrac{1}{2}m_x)^{-1}$$

$$= (1-\tfrac{1}{2}m_x)\{1-\tfrac{1}{2}m_x+\tfrac{1}{4}(m_x)^2-\tfrac{1}{8}(m_x)^3+\ldots\}$$

$$= 1-m_x+\tfrac{1}{2}(m_x)^2-\tfrac{1}{4}(m_x)^3+\ldots$$

Also from *Text Book* formula (9)

$$q_x = \frac{m_x}{1+\frac{1}{2}m_x}$$

$$= m_x(1+\tfrac{1}{2}m_x)^{-1}$$

$$= m_x\{1-\tfrac{1}{2}m_x+\tfrac{1}{4}(m_x)^2-\tfrac{1}{8}(m_x)^3+\ldots\}$$

$$= m_x-\tfrac{1}{2}(m_x)^2+\tfrac{1}{4}(m_x)^3-\tfrac{1}{8}(m_x)^4+\ldots$$

The latter result might have been derived directly from the former, since $q_x = 1 - p_x$.

3. Given the following particulars

x	q_x
20	·00572
21	·00608
22	·00643
23	·00668
24	·00691

find how many of 10,000 people living at age twenty die during each year of age up to twenty-five.

Here we are given l_{20} and q_{20}, and hence we may find d_{20}, since from *Text Book* formula (5), $l_{20} \times q_{20} = d_{20}$. Also from *Text Book* formula (1), $l_{21} = l_{20} - d_{20}$, and being given q_{21}, we may similarly find d_{21}, and so on for d_{22}, d_{23}, and d_{24}. In the example, $d_{20} = 57$, $d_{21} = 60$, $d_{22} = 64$, $d_{23} = 66$, and $d_{24} = 67$.

4. Find, out of 30,000 persons living at age thirty-five, the number who are still alive at each age up to forty, being given $m_{35} = ·00865$, $m_{36} = ·00889$, $m_{37} = ·00914$, $m_{38} = ·00942$, $m_{39} = ·00975$.

Here the first step is to find p_{35}, p_{36}, etc., from m_{35}, m_{36}, etc. By *Text Book* formula (8)

$$p_x = \frac{2 - m_x}{2 + m_x}$$

$$\text{hence} \quad p_{35} = \frac{2 - ·00865}{2 + ·00865} = ·99139$$

Also, $p_{36} = ·99115$, $p_{37} = ·99090$, $p_{38} = ·99062$, $p_{39} = ·99030$.

Now, from *Text Book* formula (4), we have $l_{x+1} = l_x p_x$, and therefore

$$\begin{aligned}
l_{36} &= l_{35}\, p_{35} \\
&= 30000 \times ·99139 \\
&= 29742 \\
l_{37} &= l_{36}\, p_{36} \\
&= 29742 \times ·99115 \\
&= 29479 \\
l_{38} &= 29211 \\
l_{39} &= 28937 \\
l_{40} &= 28656
\end{aligned}$$

CHAPTER II

Probabilities of Life

1. It is very important that the student should have at his finger-ends the values of all probabilities in which two lives may be involved, and for that purpose he should practise, till he attains complete proficiency, writing down the values of the following, giving in addition the symbols, where these are possible. The answers should be carefully compared with those given in the *Text Book*.

The probability that:—

1. (x) will survive n years.
2. (x) and (y) will both survive n years.
3. Neither (x) nor (y) will survive n years.
4. At least one of the lives (x) and (y) will survive n years.
5. (x) will survive n years and (y) die within n years.
6. Exactly one of the lives (x) and (y) will survive n years.
7. At least one of the lives (x) and (y) will fail within n years.
8. Both (x) and (y) will die in the nth year from the present time.
9. The first death will happen in the nth year from the present time.
10. The second death will happen in the nth year from the present time.
11. One only of the two lives will fail in the nth year.
12. Neither of the two lives will fail in the nth year.
13. One at least of the two lives will fail in the nth year.
14. (x) will survive n years and (y) will survive $(n-1)$ years.

With regard to the last of these, it is useful to note that, besides the form given in the *Text Book*, this probability may be written :—

$$ {}_np_x \times {}_{n-1}p_y = p_x \times {}_{n-1}p_{x+1} \times {}_{n-1}p_y = p_x \times {}_{n-1}p_{x+1:y} $$

2. To find the probability that r at least of m lives will survive n years.

An alternative proof of the formula

$$ {}_np_{\overline{xyz\ldots(m)}}^{\;r} = \frac{Z^r}{(1+Z)^r} $$

is as follows:—

This probability is equal to the sum of the probabilities that exactly r, exactly $(r+1)$, exactly $(r+2)$, and so on *ad inf.*, will survive n years (though, when $\overline{r+k} > m$, the individual probabilities will have no value).

But by *Text Book* formula (14), we have

$$ {}_np_{\overline{xyz\ldots(m)}}^{\;[r]} = \frac{Z^r}{(1+Z)^{r+1}} $$

Hence

$$ \begin{aligned} {}_np_{\overline{xyz\ldots(m)}}^{\;r} &= \frac{Z^r}{(1+Z)^{r+1}} + \frac{Z^{r+1}}{(1+Z)^{r+2}} + \frac{Z^{r+2}}{(1+Z)^{r+3}} + \ldots \\ &= \frac{Z^r}{(1+Z)^{r+1}}\left\{1 + \frac{Z}{(1+Z)} + \frac{Z^2}{(1+Z)^2} + \ldots\right\} \\ &= \frac{Z^r}{(1+Z)^{r+1}}\left(1 - \frac{Z}{1+Z}\right)^{-1} \\ &= \frac{Z^r}{(1+Z)^r} \end{aligned} $$

3. In expansion of *Text Book*, Articles 34 and 35, it may be asked what are the expected deaths and expected claims respectively amongst m joint-life policies on (x) and (y) for £K each.

The expected deaths are $m(q_x+q_y)$, and the expected claims £K$m(1-p_{xy})$. In the former of these expressions, however, it might fairly be argued that it is incompetent to take account of a second death on any one policy in the year, as the lives are not traced beyond the first death.

But in last-survivor policies this does not hold, and the expected deaths are as before $m(q_x+q_y)$, while the expected claims are £K$m(q_x \times q_y)$.

In contingent insurances payable if (x) die before (y), the expected deaths may be considered to be mq_x, and the expected claims are £K$mq_x(1-\frac{1}{2}q_y)$.

4. It is most necessary that a clear perception should be obtained of the nature of the force of mortality, and the following

explanation is offered in the belief that it will assist towards the attainment of that object.

On page 495 of the *Text Book* will be found a table of q_x, the rate of mortality. This is the probability that a person aged x will die within a year, and it is deduced from the elementary equation $q_x = \frac{d_x}{l_x}$.

It must, however, be evident on consideration that the rate of mortality is not constant between ages x and $(x+1)$, then suddenly rising to q_{x+1}; nor constant between ages $(x+1)$ and $(x+2)$, then suddenly rising to q_{x+2}, and so on. The probability that a person of any age will die within a year obviously depends upon the number who are alive at that age (whether the age may be expressed by an integer or not), and the deaths within one year after that particular age.

Now it is frequently necessary in actuarial work to have the probability that a person aged x will die at a particular moment. The function, however, that is tabulated is the force of mortality at age x, which is clearly defined in *Text Book,* Article 38, as "the proportion of persons of that age who would die in a year, if the intensity of mortality remained constant for a year, and if the number of persons under observation also remained constant, the places of those who die being constantly occupied by fresh lives."

It may be useful at this point to introduce an illustration to assist in making the idea clear. Let us consider the speed or the "force" of a railway train. This is generally measured by the distance covered in the course of an hour, *e.g.,* the speed is 40 miles an hour. Any other function if tabulated would convey but little meaning. For the same reason the force of mortality is always measured as within one year.

Suppose now we wish to measure the rate at which a train is travelling at any particular point. We might ascertain precisely the distance covered during the following minute, when simple proportion would give us the distance covered in an hour. It is obvious, however, that a minute is too long a period within which to measure; that, in fact, the rate at which the train was travelling may have varied considerably within that interval. A better result would be obtained were we to measure the distance covered during the following second, and resort as before to simple proportion. In other words, the smaller the interval of time within

which we measure, the more accurately shall we be able to gauge the rate at which the train is travelling at any particular point. It will be noticed that our answer gives us the distance that the train would cover during one hour, were the speed at which the train was travelling during the infinitely small interval of measurement to remain constant for an hour.

When now we come to measure the force of mortality at any age x, we might work out the probability that a person of that age will die within one day. Multiplying the result by 365, we should get an approximation to the force of mortality. In symbols

$$\mu_x = 365 \frac{l_x - l_{x+\frac{1}{365}}}{l_x} \text{ approximately.}$$

A day, however, is too long a period within which to measure. A better result would be obtained were we to reduce the interval to one hour. This would give us

$$\mu_x = 24 \times 365 \frac{l_x - l_{x+\frac{1}{24\times 365}}}{l_x} \text{ approximately.}$$

The smaller the interval within which we measure, the more accurate will be our result. Hence we say that

$$\mu_x = \frac{1}{t}\frac{l_x - l_{x+t}}{l_x}$$

where t approaches the limit 0.

When t approaches the limit 0, we have $dl_x = l_{x+t} - l_x$, and t, the infinitely small increase in x, is written dx. We therefore have

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx}$$

where $\frac{dl_x}{dx}$ is the first differential coefficient of l_x with respect to x.

It may be pointed out here that the value of the force of mortality among lives assured varies between zero and infinity. The value is nearly zero in the case of lives of age at entry x who have just passed the medical examination for life assurance. It is infinitely great when we come towards the end of the mortality table, say when there is only one person alive, and that one about to die. In the last year it rises from a fraction less than unity to

infinity. It may be noted that the *rate* of mortality can never exceed unity.

If the column l_x followed a mathematical law, it would be a simple matter to evaluate $\frac{dl_x}{dx}$, and hence μ_x. The several formulas that have been suggested for l_x will be discussed later in Chapter VI. Meantime we must take it that the column l_x does not follow a mathematical law, and be content to obtain an approximate formula for μ_x.

At this point we find it useful to resort to the method of Central Differences. The ordinary formula of Finite Differences for interpolation between two of a number of given values of a function is

$$u_{x+t} = u_x + t\Delta u_x + \frac{t(t-1)}{\underline{|2}}\Delta^2 u_x + \frac{t(t-1)(t-2)}{\underline{|3}}\Delta^3 u_x + \cdots$$

where all the known values but one are on the same side of the unknown value, for t is supposed to lie between 0 and 1. In a scheme for Central Differences we choose values of the function which are distributed more nearly equally on each side of the required value. We have the following

$$\begin{array}{llll}
u_{x-1} & & & \\
 & a_{x-1} & & \\
u_x & (a_x) & b_x & \\
 & a_{x+1} & & c_{x+1} \\
u_{x+1} & & b_{x+1} & \\
 & a_{x+2} & & \\
u_{x+2} & & &
\end{array}$$

$$\begin{aligned}
\text{where } a_x &= \frac{a_{x-1} + a_{x+1}}{2} \\
&= \frac{a_{x-1} + a_{x-1} + b_x}{2} \\
&= a_{x-1} + \tfrac{1}{2}b_x \\
\text{and } a_{x-1} &= a_x - \tfrac{1}{2}b_x \\
\text{Also, } u_{x-1} &= u_x - a_{x-1} \\
&= u_x - a_x + \tfrac{1}{2}b_x.
\end{aligned}$$

Then since

$$u_{x+t} = u_{x-1} + (t+1)a_{x-1} + \frac{(t+1)t}{\underline{|2}} b_x + \frac{(t+1)t(t-1)}{\underline{|3}} c_{x+1} + \cdot \cdot$$

neglecting differences beyond the second, and substituting for u_{x-1} and a_{x-1} their values as found above, we have

$$u_{x+t} = (u_x - a_x + \tfrac{1}{2}b_x) + (t+1)(a_x - \tfrac{1}{2}b_x) + \frac{(t+1)t}{2} b_x$$

$$= u_x + ta_x + \frac{t^2}{2} b_x$$

Adopting now the notation of the mortality table, and writing l_x for u_x, we have

$$l_{x+t} = l_x + ta_x + \frac{t^2}{2} b_x$$

$$\text{whence} \quad \frac{l_{x+t} - l_x}{t} = a_x + \frac{t}{2} b_x$$

from which we get, in the limit when t approaches 0,

$$\frac{dl_x}{dx} = a_x$$

$$= \frac{a_{x-1} + a_{x+1}}{2}$$

$$= \frac{l_{x+1} - l_{x-1}}{2} \text{ or } - \left(\frac{d_{x-1} + d_x}{2}\right)$$

$$\text{But} \quad \mu_x = -\frac{1}{l_x} \frac{dl_x}{dx}$$

$$\text{Therefore,} \quad \mu_x = -\frac{1}{l_x} \frac{l_{x+1} - l_{x-1}}{2} = \frac{l_{x-1} - l_{x+1}}{2l_x}$$

$$\text{or} \quad \mu_x = -\frac{1}{l_x} \times -\left(\frac{d_{x-1} + d_x}{2}\right) = \frac{d_{x-1} + d_x}{2l_x}$$

We might otherwise arrive at the same result by a process slightly different.

$$\frac{du_x}{dx} = \frac{u_{x+t} - u_x}{t} \text{ when } t \text{ approaches the limit 0,}$$

$$= \frac{\left\{u_x + t\Delta u_x + \frac{t(t-1)}{\underline{|2}}\Delta^2 u_x + \frac{t(t-1)(t-2)}{\underline{|3}}\Delta^3 u_x + \cdot \cdot \cdot \right\} - u_x}{t}$$

when t approaches the limit 0,

$$\frac{du_x}{dx} = \Delta u_x + \frac{(t-1)}{2}\Delta^2 u_x + \frac{(t-1)(t-2)}{6}\Delta^3 u_x + \cdots$$

when t approaches the limit 0,

$$= \Delta u_x - \tfrac{1}{2}\Delta^2 u_x + \tfrac{1}{3}\Delta^3 u_x - \cdots$$

$$= \Delta u_x - \tfrac{1}{2}\Delta^2 u_x \text{ approximately.}$$

Hence, since in this approximation third and higher differences are held to vanish, and therefore second differences are constant,

$$\frac{du_x}{dx} = \Delta u_x - \tfrac{1}{2}\Delta^2 u_{x-1} \text{ approximately,}$$

$$= \Delta u_x - \tfrac{1}{2}(\Delta u_x - \Delta u_{x-1})$$

$$= \tfrac{1}{2}(\Delta u_x + \Delta u_{x-1})$$

$$= \frac{u_{x+1} - u_{x-1}}{2} \text{ approximately.}$$

From this we get as before

$$\frac{dl_x}{dx} = \frac{l_{x+1} - l_{x-1}}{2}$$

$$\text{and } \mu_x = \frac{l_{x-1} - l_{x+1}}{2l_x}$$

This formula provides a good working approximation to the value of the force of mortality in almost all cases.

Again arguing from the same formula as the preceding result was obtained from, we may obtain a general formula for μ_x.

We have successively

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{1}{l_x}(\Delta l_x - \tfrac{1}{2}\Delta^2 l_x + \tfrac{1}{3}\Delta^3 l_x - \cdots)$$

$$= -\frac{1}{l_x}(-d_x) \text{ (stopping at first differences)}$$

$$= q_x$$

$$\mu_x = -\frac{1}{l_x}\{-d_x - \tfrac{1}{2}(d_x - d_{x+1})\} \text{ (stopping at second differences)}$$

$$= q_x + \tfrac{1}{2}(q_x - {}_1|q_x)$$

$$= q_x(1 + \tfrac{1}{2}) - \tfrac{1}{2} \times {}_1|q_x$$

$$\mu_x = -\frac{1}{l_x}\{-d_x - \tfrac{1}{2}(d_x - d_{x+1}) - \tfrac{1}{3}(d_x - 2d_{x+1} + d_{x+2})\}$$

(stopping at third differences)

$$\mu_x = q_x + \tfrac{1}{2}(q_x - {}_{1|}q_x) + \tfrac{1}{3}(q_x - 2 \times {}_{1|}q_x + {}_{2|}q_x)$$
$$= q_x(1 + \tfrac{1}{2} + \tfrac{1}{3}) - {}_{1|}q_x(\tfrac{1}{2} + \tfrac{2}{3}) + \tfrac{1}{3} \times {}_{2|}q_x$$
$$\mu_x = q_x(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) - {}_{1|}q_x(\tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4}) + {}_{2|}q_x(\tfrac{1}{3} + \tfrac{3}{4}) - \tfrac{1}{4} \times {}_{3|}q_x$$

(stopping at fourth differences)

And generally

$$\mu_x = q_x(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots) - {}_{1|}q_x(\tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \tfrac{4}{5} + \cdots)$$
$$+ {}_{2|}q_x(\tfrac{1}{3} + \tfrac{3}{4} + \tfrac{6}{5} + \cdots) - {}_{3|}q_x(\tfrac{1}{4} + \tfrac{4}{5} + \cdots)$$
$$+ {}_{4|}q_x(\tfrac{1}{5} + \cdots) - \text{ etc.}$$

The proof that the central death-rate is equal to the value of the force of mortality at an age half a year older is as follows:—

Making the usual assumption of a uniform distribution of deaths throughout each year of age, we have

$$(l_{x-1} - l_x) = d_{x-1}$$
$$= 2(l_{x-\frac{1}{2}} - l_x)$$

and
$$(l_x - l_{x+1}) = d_x$$
$$= 2(l_x - l_{x+\frac{1}{2}})$$

But
$$\mu_x = \frac{l_{x-1} - l_{x+1}}{2l_x}$$
$$= \frac{(l_{x-1} - l_x) + (l_x - l_{x+1})}{2l_x}$$
$$= \frac{2(l_{x-\frac{1}{2}} - l_x) + 2(l_x - l_{x+\frac{1}{2}})}{2l_x}$$
$$= \frac{l_{x-\frac{1}{2}} - l_{x+\frac{1}{2}}}{l_x}$$
$$= \frac{d_{x-\frac{1}{2}}}{l_x}$$
$$= m_{x-\frac{1}{2}}$$

whence also $m_x = \mu_{x+\frac{1}{2}}$

EXAMPLES

1. Write down formulas for all the possible combinations of the probability of dying in or surviving a year among three lives, and prove the truth of your answer.

(x)	(y)	(z)		
live	live	live	=	p_{xyz}
live	live	die	=	$p_{xy}(1-p_z)$
live	die	live	=	$p_{xz}(1-p_y)$
die	live	live	=	$p_{yz}(1-p_x)$
live	die	die	=	$p_x(1-p_y)(1-p_z)$
die	live	die	=	$p_y(1-p_x)(1-p_z)$
die	die	live	=	$p_z(1-p_x)(1-p_y)$
die	die	die	=	$(1-p_x)(1-p_y)(1-p_z)$

If these probabilities be summed, the result will be found to be unity, and thus proof is obtained that all possible contingencies have been noted.

2. The probability that two persons aged respectively twenty and forty will not both be alive at the end of 20 years is ·38823. Out of 96,223 persons alive at age twenty, 6,358 die before they attain age thirty. Find $|_{30}q_{30}$.

$$
\begin{aligned}
|_{30}q_{30} &= 1 - {}_{30}p_{30} \\
&= 1 - \frac{l_{60}}{l_{30}} \\
&= 1 - \frac{l_{60}}{l_{40}}\frac{l_{40}}{l_{20}}\frac{l_{20}}{l_{30}} \\
&= 1 - (1 - \cdot 38823) \times \frac{96223}{89865} \\
&= \cdot 34495
\end{aligned}
$$

3. Given that the probability that two persons aged twenty-five and fifty respectively will both live 25 years is ·27516, and that, by the same mortality table, out of 93,044 persons alive at age twenty-five, 82,277 attain age forty, what is the probability

that a person aged forty will survive till the attainment of age seventy-five?

$$
\begin{aligned}
{}_{35}p_{40} &= \frac{l_{75}}{l_{40}} = \frac{l_{75}}{l_{50}}\frac{l_{50}}{l_{25}}\frac{l_{25}}{l_{40}} \\
&= \frac{l_{50:75}}{l_{25:50}}\frac{l_{25}}{l_{40}} = {}_{25}p_{25:50} \times \frac{l_{25}}{l_{40}} \\
&= \cdot 27516 \times \frac{93044}{82277} \\
&= \cdot 31117
\end{aligned}
$$

4. An annuity society is formed in which members may secure an annuity of m at age $x+n$ by payment of a single sum at age x. If k members aged x start the society and l new members of the same age join each subsequent year, find how many members will be entitled to rank for annuities at the end of $n+t$ years and the corresponding amount payable.

Of l_x entering now l_{x+n+t} will survive at the end of $n+t$ years, and therefore of k entering now $k\,\frac{l_{x+n+t}}{l_x}$ will survive at the end of $n+t$ years. Again, of l_x entering one year hence $l_{x+n+t-1}$ will survive at the end of $n+t$ years from now, and of l entering one year hence $l\,\frac{l_{x+n+t-1}}{l_x}$ will survive; and so on for succeeding years.

Thus the total number surviving at the end of $n+t$ years and entitled to rank for annuities will be

$$
k\,\frac{l_{x+n+t}}{l_x} + l\left(\frac{l_{x+n+t-1}}{l_x} + \frac{l_{x+n+t-2}}{l_x} + \cdots + \frac{l_{x+n}}{l_x}\right)
$$

Each of these gets an annuity of m, and therefore the total amount of annuities in force will be

$$
m\{k\,{}_{n+t}p_x + l({}_{n+t-1}p_x + {}_{n+t-2}p_x + \cdots + {}_np_x)\}
$$

5. Obtain from the *Text Book* mortality table the numerical values of the probability that out of three lives 30, 35, and 40

(1) One, at least, will die in the 10th year.

(2) Not more than two will fail in the 10th year.

(3) All will die within 20 years.

(1) This probability in symbols is

$$1-(1-{}_{9|}q_{30})(1-{}_{9|}q_{35})(1-{}_{9|}q_{40})$$
$$=1-\left(1-\frac{d_{39}}{l_{30}}\right)\left(1-\frac{d_{44}}{l_{35}}\right)\left(1-\frac{d_{49}}{l_{40}}\right)$$
$$=1-\left(1-\frac{806}{89685}\right)\left(1-\frac{924}{86137}\right)\left(1-\frac{1101}{82277}\right)$$
$$=1-\frac{88879}{89685}\times\frac{85213}{86137}\times\frac{81176}{82277}$$
$$=\cdot 03274.$$

(2) This is the probability that all three will not die in the 10th year, and is equal to

$$1-{}_{9|}q_{30}\times{}_{9|}q_{35}\times{}_{9|}q_{40}$$
$$=1-\frac{d_{39}}{l_{30}}\times\frac{d_{44}}{l_{35}}\times\frac{d_{49}}{l_{40}}$$
$$=1-\frac{806}{89685}\times\frac{924}{86137}\times\frac{1101}{82277}$$
$$=\cdot 9999987.$$

(3) $$|_{20}q_{\overline{30:35:40}} = (1-{}_{20}p_{30})(1-{}_{20}p_{35})(1-{}_{20}p_{40})$$
$$=\frac{l_{30}-l_{50}}{l_{30}}\times\frac{l_{35}-l_{55}}{l_{35}}\times\frac{l_{40}-l_{60}}{l_{40}}$$
$$=\frac{16890}{89685}\times\frac{19571}{86137}\times\frac{23435}{82277}$$
$$=\cdot 01219.$$

6. There are X persons living aged x, and the number of combinations of them taken 3 together is 35. What is the probability that, at the end of n years, the number of combinations of the survivors taken 3 together, will be at least 10?

By inspection one may see that the number of combinations of seven persons taken 3 together is 35, and of five taken 3 together is 10. Thus the question is to find the probability that at least five persons out of seven of age x will survive n years.

Now $${}_np_{\overline{xyz\ldots(m)}}^{\,r} = \frac{Z^r}{(1+Z)^r}$$

Therefore, $${}_np_{\overline{xxxxxxx}}^{\,5} = \frac{Z^5}{(1+Z)^5}$$
$$= Z^5-5Z^6+15Z^7$$
$$= {}_7C_5({}_np_x)^5-5\,{}_7C_6({}_np_x)^6+15({}_np_x)^7$$

7. Find the probability that, out of five lives all aged x, one designated life, A, will die in the year and be the first to die.

This may happen in several different ways,

(1) A alone may die in the year, the other four surviving to the end of the year. The probability of this event happening is

$$q_x(p_x)^4.$$

(2) A and one other may die in the year (A first), and the other three survive to the end of the year. This probability is

$$_4C_1 \tfrac{1}{2}(q_x)^2(p_x)^3.$$

(3) A and two others (A first), the probability being

$$_4C_2 \tfrac{1}{3}(q_x)^3(p_x)^2.$$

(4) A and three others (A first),

$$_4C_3 \tfrac{1}{4}(q_x)^4 p_x$$

(5) All (A first), $\tfrac{1}{5}(q_x)^5.$

The total probability therefore is

$$\begin{aligned} & q_x(p_x)^4 + {}_4C_1 \tfrac{1}{2}(q_x)^2(p_x)^3 + {}_4C_2 \tfrac{1}{3}(q_x)^3(p_x)^2 \\ & \quad + {}_4C_3 \tfrac{1}{4}(q_x)^4 p_x + \tfrac{1}{5}(q_x)^5 \\ & = \tfrac{1}{5}[\{(p_x)^5 + 5q_x(p_x)^4 + 10(q_x)^2(p_x)^3 + 10(q_x)^3(p_x)^2 \\ & \quad + 5(q_x)^4 p_x + (q_x)^5\} - (p_x)^5] \\ & = \tfrac{1}{5}\{(p_x + q_x)^5 - (p_x)^5\} \\ & = \tfrac{1}{5}\{1 - (p_x)^5\} \end{aligned}$$

8. Find expressions for the following probabilities :—

That out of 25 persons aged x,

(*a*) Exactly 5 will die in a year.

(*b*) Not more than 5 will die in a year

(*c*) 5 designated individuals and no more will die in a year.

(*d*) 5 designated individuals at least will die in a year.

(*a*) The 5 may be chosen in $_{25}C_5$ ways, and therefore the probability is $_{25}C_5(q_x)^5(p_x)^{20}$.

(*b*) This is the sum of the probabilities that none, exactly 1, exactly 2, exactly 3, exactly 4, and exactly 5 will die, and is therefore equal to

$$\begin{aligned} & (p_x)^{25} + {}_{25}C_1 q_x(p_x)^{24} + {}_{25}C_2(q_x)^2(p_x)^{23} + {}_{25}C_3(q_x)^3(p_x)^{22} \\ & \quad + {}_{25}C_4(q_x)^4(p_x)^{21} + {}_{25}C_5(q_x)^5(p_x)^{20}. \end{aligned}$$

(c) The 5 can only be chosen in one way, and therefore the probability is $(q_x)^5(p_x)^{20}$.

(d) It is no matter what happens to the remaining 20, and therefore the probability is simply $(q_x)^5$.

9. Two offices have each £1,000,000 assured — office A, by 100 policies of £10,000 each; B, by 1000 of £1000 each. Assuming all the ages equal, and the rate of mortality to be 2 per cent. per annum, give an expression for the probability in each case that the claims will amount in one year to £30,000 at least.

In the case of office A, in order that not less than £30,000 of claims should be made, not less than 3 deaths in 100 must occur. In other words, the deaths must not number either 0, 1, or 2. The probability required is therefore

$$1-\left\{\left(\frac{49}{50}\right)^{100}+{}_{100}C_1\left(\frac{49}{50}\right)^{99}\frac{1}{50}+{}_{100}C_2\left(\frac{49}{50}\right)^{98}\left(\frac{1}{50}\right)^2\right\},$$

the probability of one life's surviving being, according to data, $\frac{98}{100}$ or $\frac{49}{50}$, and of its dying $\frac{1}{50}$.

In office B, the deaths must number not less than 30 in 1000, if £30,000 at least is to be claimed; and the probability of this happening is, similarly to the above,

$$1-\left\{\left(\frac{49}{50}\right)^{1000}+{}_{1000}C_1\left(\frac{49}{50}\right)^{999}\frac{1}{50}+\cdot\cdot\cdot+{}_{1000}C_{29}\left(\frac{49}{50}\right)^{971}\left(\frac{1}{50}\right)^{29}\right\}$$

the probabilities of surviving and dying being as above.

10. At the beginning of a year there are 100 policies in force, each for £100, and each payable on the death of the survivor of two lives, the ages in each case being thirty-six and fifty-five respectively:—Calculate (1) the expected mortality in the year; and (2) the expected claims in the year—given $p_{36}=\cdot991$, and $p_{55}=\cdot980$.

(1) The expected mortality is

$$100(q_{36}+q_{55}) = 100(\cdot009+\cdot020) = 2\cdot9.$$

(2) The expected claims are

$$£10000(q_{36}\times q_{55}) = £10000(\cdot009\times\cdot020) = £1\cdot8.$$

11. Show approximately that the force of mortality at age x is greater than the probability of dying in the year after age x, when the number dying in the year after age $x-1$ is greater than the number so dying after age x.

$$\mu_x > = < q_x$$

approximately as

$$\frac{d_{x-1}+d_x}{2l_x} > = < \frac{d_x}{l_x}$$

according as

$$d_{x-1}+d_x > = < 2d_x$$

that is, according as

$$d_{x-1} > = < d_x$$

12. Show that $$q_x = \frac{1}{l_x}\int_0^1 l_{x+t}\mu_{x+t}\,dt$$

$$\begin{aligned}\frac{1}{l_x}\int_0^1 l_{x+t}\mu_{x+t}dt &= \frac{1}{l_x}\int_0^1 l_{x+t}\left(-\frac{1}{l_{x+t}}\frac{dl_{x+t}}{dt}\right)dt\\ &= -\frac{1}{l_x}\int_0^1 \frac{dl_{x+t}}{dt}dt\\ &= -\frac{1}{l_x}(l_{x+1}-l_x)\\ &= \frac{d_x}{l_x}\\ &= q_x\end{aligned}$$

13. Prove that $$l_x = \int_0^\infty l_{x+t}\,\mu_{x+t}\,dt$$

$$\begin{aligned}\int_0^\infty l_{x+t}\mu_{x+t}dt &= -\int_0^\infty l_{x+t}\frac{1}{l_{x+t}}\frac{dl_{x+t}}{dt}dt\\ &= -\int_0^\infty \frac{dl_{x+t}}{dt}dt\\ &= -(l_{x+\infty}-l_x)\\ &= l_x\end{aligned}$$

14. Show that approximately $\operatorname{colog}_e p_x = \mu_{x+\frac{1}{2}} + \frac{1}{12}(q_x)^3$.

$$\begin{aligned}\operatorname{colog}_e p_x &= -\log_e p_x \\ &= -\log_e(1-q_x) \\ &= q_x + \frac{(q_x)^2}{2} + \frac{(q_x)^3}{3} + \text{ etc.} \qquad . \quad . \ (1)\end{aligned}$$

$$\begin{aligned}\text{Also } \mu_{x+\frac{1}{2}} &= m_x \text{ approximately} \\ &= \frac{q_x}{1-\frac{1}{2}q_x} \\ &= q_x(1-\tfrac{1}{2}q_x)^{-1} \\ &= q_x + \frac{(q_x)^2}{2} + \frac{(q_x)^3}{4} + \text{ etc.} \qquad . \quad . \ (2)\end{aligned}$$

Stopping at the term involving $(q_x)^3$, and deducting (2) from (1)

$$\begin{aligned}\operatorname{colog}_e p_x - \mu_{x+\frac{1}{2}} &= \frac{(q_x)^3}{12} \\ \text{and } \operatorname{colog}_e p_x &= \mu_{x+\frac{1}{2}} + \tfrac{1}{12}(q_x)^3\end{aligned}$$

15. Show how to obtain an approximation to $\mu_{[x]}$.

Here the ordinary approximate formula $\mu_x = \dfrac{l_{x-1} - l_{x+1}}{2l_x}$ fails us; for if we wrote $\mu_{[x]} = \dfrac{l_{[x]-1} - l_{[x]+1}}{2l_{[x]}}$ we could assign no meaning to $l_{[x]-1}$. The $l_{[x]}$ persons aged x are select, and come under observation at that age for the first time, and consequently we know nothing of the persons of age $x-1$, of whom they are the survivors. We must accordingly seek another approximation.

$$\begin{aligned}\mu_{[x]} &= -\frac{1}{l_{[x]}}\frac{dl_{[x]}}{dx} \\ &= -\frac{1}{l_{[x]}}(\Delta l_{[x]} - \tfrac{1}{2}\Delta^2 l_{[x]} + \tfrac{1}{3}\Delta^3 l_{[x]} - \cdot \cdot \cdot) \text{ approximately.}\end{aligned}$$

If then we take the column $l_{[x]}$, $l_{[x]+1}$, $l_{[x]+2}$, etc., and difference it, we obtain successively $\Delta l_{[x]}$, $\Delta^2 l_{[x]}$, $\Delta^3 l_{[x]}$, etc. If, further, these be divided by 1, 2, 3, etc., respectively, and the sum of the odd terms deducted from the sum of the even, the result, divided by $l_{[x]}$, will give us an approximation to $\mu_{[x]}$.

Or we may proceed thus—

$$\mu_{[x]} = -\frac{1}{l_{[x]}}\frac{dl_{[x]}}{dx}$$

$$= -\frac{d\log_e l_{[x]}}{dx}$$

$$= -\frac{1}{\mathrm{M}}\frac{d\log_{10} l_{[x]}}{dx}$$ (M being the modulus of common logarithms and equal to ·4342945).

$$= -\frac{1}{\mathrm{M}}(\Delta\log_{10} l_{[x]} - \tfrac{1}{2}\Delta^2\log_{10} l_{[x]} + \tfrac{1}{3}\Delta^3\log_{10} l_{[x]} - \cdot\ \cdot\ \cdot\ )\text{approx.}$$

Following a method similar to that indicated above we obtain another approximation to the value of $\mu_{[x]}$.

CHAPTER III

Expectations of Life

1. The definitions of the following functions and the distinctions between them should be carefully noted.

The *Complete Expectation of Life* at any age is the average future lifetime of each person of that age.

The *Curtate Expectation of Life* at any age is the average number of complete years which will be lived by each person of that age.

The expectation of life, or more properly the complete expectation of life, is also sometimes called the "mean after-lifetime"; the "average after-lifetime"; the "mean duration of life"; or the "average duration of life."

The *most probable after-lifetime* at any age is the difference between that age and the year of age in which the life will most probably fail, that is, the year in which most deaths occur.

The *Vie Probable* at any age is the difference between that age and the year of age to which there is an even chance of living, that is, the year in which the number living is reduced to one half the original.

2. In *Text Book*, Article 24, Lubbock's formula is applied to find a more exact expression for $\mathring{e}_x$ than $e_x+\frac{1}{2}$. The deduction of the formula itself may be presented as follows:—

To find the sum of the series

$$u_0+u_{\frac{1}{t}}+u_{\frac{2}{t}}+\cdot\cdot\cdot\; +u_1+u_{1+\frac{1}{t}}+u_{1+\frac{2}{t}}+\cdot\cdot\cdot$$

to the end of the mortality table, the values ultimately disappearing whatever function u may represent.

We have then

$$u_0 = u_0$$

$$u_{\frac{1}{t}} = u_0 + \frac{1}{t}\Delta u_0 + \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)}{\underline{|2}}\Delta^2 u_0 + \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)\left(\frac{1}{t}-2\right)}{\underline{|3}}\Delta^3 u_0 + \cdots$$

$$u_{\frac{2}{t}} = u_0 + \frac{2}{t}\Delta u_0 + \frac{\frac{2}{t}\left(\frac{2}{t}-1\right)}{\underline{|2}}\Delta^2 u_0 + \frac{\frac{2}{t}\left(\frac{2}{t}-1\right)\left(\frac{2}{t}-2\right)}{\underline{|3}}\Delta^3 u_0 + \cdots$$

etc. etc.

$$u_{\frac{t-1}{t}} = u_0 + \frac{t-1}{t}\Delta u_0$$

$$+ \frac{\frac{t-1}{t}\left(\frac{t-1}{t}-1\right)}{\underline{|2}}\Delta^2 u_0 + \frac{\frac{t-1}{t}\left(\frac{t-1}{t}-1\right)\left(\frac{t-1}{t}-2\right)}{\underline{|3}}\Delta^3 u_0 + \cdots$$

And summing these

$$\Sigma_0^{\frac{t-1}{t}} u = tu_0 + \frac{t-1}{2}\Delta u_0 - \frac{t^2-1}{12t}\Delta^2 u_0 + \frac{t^2-1}{24t}\Delta^3 u_0 - \cdots$$

since coefficient of Δu_0

$$= \frac{1}{t}(1+2+\cdots+\overline{t-1}) = \frac{t(t-1)}{2t} = \frac{t-1}{2}$$

coefficient of $\Delta^2 u_0$

$$= \frac{1}{2t^2}\{1^2+2^2+\cdots+(t-1)^2\} - \frac{1}{2t}(1+2+\cdots+\overline{t-1})$$

$$= \frac{(t-1)t(2t-1)}{12t^2} - \frac{t-1}{4} = -\frac{t^2-1}{12t}$$

and so on.

Similarly

$$\Sigma_1^{\frac{2t-1}{t}} u = tu_1 + \frac{t-1}{2}\Delta u_1 - \frac{t^2-1}{12t}\Delta^2 u_1 + \frac{t^2-1}{24t}\Delta^3 u_1 - \cdots$$

$$\Sigma_2^{\frac{3t-1}{t}} u = tu_2 + \frac{t-1}{2}\Delta u_2 - \frac{t^2-1}{12t}\Delta^2 u_2 + \frac{t^2-1}{24t}\Delta^3 u_2 - \cdots$$

and so on.

Now summing these summations, we get

$$u_0+u_{\frac{1}{t}}+u_{\frac{2}{t}}+\cdot\cdot\cdot+u_1+u_{1+\frac{1}{t}}+u_{1+\frac{2}{t}}+\cdot\cdot\cdot$$

$$= t(u_0+u_1+u_2+\cdot\cdot\cdot)+\frac{t-1}{2}(\Delta u_0+\Delta u_1+\Delta u_2+\cdot\cdot\cdot)$$

$$-\frac{t^2-1}{12t}(\Delta^2 u_0+\Delta^2 u_1+\Delta^2 u_2+\cdot\cdot\cdot)$$

$$+\frac{t^2-1}{24t}(\Delta^3 u_0+\Delta^3 u_1+\Delta^3 u_2+\cdot\cdot\cdot)-\text{ etc.}$$

$$= t(u_0+u_1+u_2+\cdot\cdot\cdot)-\frac{t-1}{2}u_0+\frac{t^2-1}{12t}\Delta u_0-\frac{t^2-1}{24t}\Delta^2 u_0+\text{ etc.}$$

Subtracting u_0 from both sides we have

$$u_{\frac{1}{t}}+u_{\frac{2}{t}}+\cdot\cdot\cdot+u_1+u_{1+\frac{1}{t}}+u_{1+\frac{2}{t}}+\cdot\cdot\cdot$$

$$= t(u_1+u_2+\cdot\cdot\cdot)+\frac{t-1}{2}u_0+\frac{t^2-1}{12t}\Delta u_0-\frac{t^2-1}{24t}\Delta^2 u_0+\text{ etc.}$$

For the application of this formula to the case of the complete expectation of life, we may proceed as follows:—

If we were to say

$$\mathring{e}_x = \frac{1}{l_x}(l_{x+1}+l_{x+2}+l_{x+3}+\cdot\cdot\cdot)$$

we should be wrong in that we take account of no more than complete years lived. We should therefore obtain a somewhat better result from

$$\mathring{e}_x = \frac{1}{2l_x}(l_{x+\frac{1}{2}}+l_{x+1}+l_{x+1\frac{1}{2}}+l_{x+2}+\cdot\cdot\cdot)$$

The same error in principle appears, however, for we take account only of complete half-years lived; and we shall obtain a correct result only from

$$\mathring{e}_x = \frac{1}{tl_x}(l_{x+\frac{1}{t}}+l_{x+\frac{2}{t}}+l_{x+\frac{3}{t}}+\cdot\cdot\cdot)$$

where $\frac{1}{t}$ is smaller than any assignable value.

We now use the summation formula, and say

$$\mathring{e}_x = \frac{1}{tl_x}\left\{t(l_{x+1}+l_{x+2}+\ldots)+\frac{t-1}{2}l_x+\frac{t^2-1}{12t}\Delta l_x - \frac{t^2-1}{24t}\Delta^2 l_x + \text{etc.}\right\}$$

$$= e_x + \frac{t-1}{2t} + \frac{t^2-1}{12t^2}\,\frac{\Delta l_x - \frac{1}{2}\Delta^2 l_x}{l_x} + \ldots$$

$$= \text{(where } \frac{1}{t} \text{ is infinitely small) } e_x + \tfrac{1}{2} + \tfrac{1}{12}\,\frac{\Delta l_x - \frac{1}{2}\Delta^2 l_x}{l_x} + \ldots$$

But in discussing the force of mortality in Chapter II. we showed that

$$\frac{dl_x}{dx} = \Delta l_x - \tfrac{1}{2}\Delta^2 l_x \text{ approximately.}$$

Therefore

$$\mathring{e}_x = e_x + \tfrac{1}{2} + \frac{1}{12l_x}\frac{dl_x}{dx} \text{ approximately}$$

$$= e_x + \tfrac{1}{2} - \frac{\mu_x}{12}$$

$$= e_x + \tfrac{1}{2} - \frac{l_{x-1} - l_{x+1}}{24l_x}$$

The process by which $\mathring{e}_x$ was obtained was perfectly general, and we may similarly write

$$\mathring{e}_{xy} = e_{xy} + \tfrac{1}{2} - \frac{\mu_x + \mu_y}{12}$$

and generally

$$\mathring{e}_{xyz\ldots(m)} = e_{xyz\ldots(m)} + \tfrac{1}{2} - \frac{\mu_x + \mu_y + \mu_z + \ \cdots \text{ to } m \text{ terms}}{12}$$

EXAMPLES

1. Given the following mortality table, deduce, in respect of a life aged eighty-two, (*a*) the curtate expectation of life, (*b*) the *vie probable*, (*c*) the age at which it is most probable that he will die. Find also the average age at death of the 129 lives, aged ninety-five.

x	l_x	d_x	x	l_x	d_x
82	10096	1712	92	575	209
83	8384	1540	93	366	144
84	6844	1361	94	222	93
85	5483	1180	95	129	58
86	4303	1002	96	71	34
87	3301	830	97	37	18
88	2471	671	98	19	10
89	1800	527	99	9	5
90	1273	402	100	4	3
91	871	296	101	1	1

Answers: (*a*) 3·582.9.
(*b*) 3·5 approximately.
(*c*) 82.

The average age at death of those aged ninety-five will be 96·593.

2. From the *Text Book* mortality table ascertain the values of $|_{10}\mathring{e}_{30}$ and $_5|_{15}\mathring{e}_{40}$.

$$|_{10}\mathring{e}_{30} = 9{\cdot}599, \quad {}_5|_{15}\mathring{e}_{40} = 12{\cdot}643.$$

3. Deduce a formula for $|_n\mathring{e}_x$ without making the assumption of a uniform distribution of deaths.

By *Text Book* formula (27) we have approximately

$$\mathring{e}_x = e_x + \tfrac{1}{2} - \tfrac{1}{12}\mu_x$$

Now
$$\begin{aligned} |_n\mathring{e}_x &= \mathring{e}_x - {}_np_x\mathring{e}_{x+n} \\ &= (e_x + \tfrac{1}{2} - \tfrac{1}{12}\mu_x) - {}_np_x(e_{x+n} + \tfrac{1}{2} - \tfrac{1}{12}\mu_{x+n}) \\ &= |_ne_x + \tfrac{1}{2}(1 - {}_np_x) - \tfrac{1}{12}(\mu_x - {}_np_x\mu_{x+n}) \end{aligned}$$

CHAPTER IV

Probabilities of Survivorship

1. In *Text Book*, Article 3, Q^1_{xy} is derived from formula (1) by giving to n successively every integral value from unity upwards. It may be derived by a similar process from formula (2)

$$_{n-1|}q^1_{xy} = \frac{d_{x+n-1} \times l_{y+n-\frac{1}{2}}}{l_{xy}}$$

Here then
$$Q^1_{xy} = \Sigma \frac{d_{x+n-1} \times l_{y+n-\frac{1}{2}}}{l_x \times l_y}$$
$$= \Sigma \frac{l_{x+n-1} - l_{x+n}}{l_x} \times \frac{l_{y+n-1} + l_{y+n}}{2l_y}$$
$$= \Sigma \frac{1}{2}\left(\frac{l_{x+n-1}\, l_{y+n-1}}{l_x l_y} - \frac{l_{x+n}\, l_{y+n-1}}{l_x l_y} + \frac{l_{x+n-1}\, l_{y+n}}{l_x l_y} - \frac{l_{x+n}\, l_{y+n}}{l_x l_y}\right)$$
$$= \frac{1}{2}\left(1 - \frac{e_{x:y-1}}{p_{y-1}} + \frac{e_{x-1:y}}{p_{x-1}}\right)$$

since
$$\Sigma\left(\frac{l_{x+n-1}\, l_{y+n-1}}{l_x l_y} - \frac{l_{x+n}\, l_{y+n}}{l_x l_y}\right) = 1$$

2. The formula $Q^1_{xy} = \frac{1}{2}\left(1 - \frac{e_{x:y-1}}{p_{y-1}} + \frac{e_{x-1:y}}{p_{x-1}}\right)$ is, by the introduction of ages $x-1$ and $y-1$, in a form unsuitable for application to select tables. To render it in a suitable form we have

$$Q^1_{xy} = \frac{1}{2}\left(1 - \frac{e_{x:y-1}}{p_{y-1}} + \frac{e_{x-1:y}}{p_{x-1}}\right)$$
$$= \frac{1}{2}\left\{1 - \frac{p_{x:y-1}(1 + e_{x+1:y})}{p_{y-1}} + \frac{p_{x-1:y}(1 + e_{x:y+1})}{p_{x-1}}\right\}$$

since $e_{x:y-1} = p_{x:y-1}(1+e_{x+1:y})$ and $e_{x-1:y} = p_{x-1:y}(1+e_{x:y+1})$

$$Q^1_{xy} = \frac{1}{2}\left\{1 - p_x(1+e_{x+1:y}) + p_y(1+e_{x:y+1})\right\}$$

3. By the use of *Text Book* formula (2) we may also very easily arrive at the probability that (x) will die before (y) or within t years after the death of (y), *i.e.*, formula (14), as follows:—

$$\text{We have}\quad Q^1_{x:\overline{y(t|)}} = (1 - {}_tp_x) + \frac{\Sigma d_{x+t+n-1}\, l_{y+n-\frac{1}{2}}}{l_x l_y}$$

$$= (1 - {}_tp_x) + \frac{l_{x+t}}{l_x}\cdot\frac{\Sigma d_{x+t+n-1}\, l_{y+n-\frac{1}{2}}}{l_{x+t}\, l_y}$$

$$= (1 - {}_tp_x) + {}_tp_x Q_{\overset{1}{x+t}:y}$$

$$= 1 - {}_tp_x(1 - Q_{\overset{1}{x+t}:y})$$

4. To find the more restricted probability that (x) will die within t years after the death of (y), we have the required probability

$$= \frac{\Sigma d_{y+n-1}(l_{x+n-\frac{1}{2}} - l_{x+n+t-\frac{1}{2}})}{l_y l_x}$$

$$= \frac{\Sigma d_{y+n-1}\, l_{x+n-\frac{1}{2}}}{l_y l_x} - \frac{l_{x+t}}{l_x}\,\frac{\Sigma d_{y+n-1}\, l_{x+n+t-\frac{1}{2}}}{l_y\, l_{x+t}}$$

$$= Q_{xy}^{\;1} - {}_tp_x Q_{x+t:\overset{1}{y}}$$

$$= (1 - Q^1_{xy}) - {}_tp_x(1 - Q_{\overset{1}{x+t}:y})$$

$$= 1 - {}_tp_x(1 - Q_{\overset{1}{x+t}:y}) - Q^1_{xy}$$

$$= Q^1_{x:\overline{y(t|)}} - Q^1_{xy}$$

which is obviously correct; for, if we take the probability that (x) will die before (y) from the probability that (x) will die before (y) or within t years after, we are left with the probability that (x) will die within t years after the death of (y).

5. The allied probability that (x) will be alive t years after the death of (y) is found as follows:—

Taking the nth year, the probability of (y) dying therein is $\dfrac{d_{y+n-1}}{l_y}$, and the probability of (x) being alive at the end of

t years after the middle of the nth year (since (y)'s death will occur at the middle of the year on the average) is $\frac{l_{x+n+t-\frac{1}{2}}}{l_x}$. Hence the required probability

$$= \frac{\Sigma d_{y+n-1}\, l_{x+n+t-\frac{1}{2}}}{l_y\, l_x}$$

$$= {}_tp_x \mathrm{Q}_{x+t\,:\,\overset{1}{y}} \text{ as above.}$$

This probability is the same as the probability that (x) will *not* die before (y) or within t years after (y)'s death, since

$$\begin{aligned} {}_tp_x \mathrm{Q}_{x+t\,:\,\overset{1}{y}} &= {}_tp_x(1 - \mathrm{Q}_{\overset{1}{x+t}\,:\,y}) \\ &= 1 - \{1 - {}_tp_x(1 - \mathrm{Q}_{\overset{1}{x+t}\,:\,y})\} \\ &= 1 - \mathrm{Q}^1_{x\,:\,\overline{y(\overline{t}|)}} \end{aligned}$$

6. The preceding probability must be clearly distinguished from the probability that (x) will be alive at the end of the tth year succeeding that in which (y) dies, which may be found thus—

Taking the nth year, the probability that (y) will die therein is $\frac{d_{y+n-1}}{l_y}$, and the probability that (x) will live to the end of the tth after the nth year is $\frac{l_{x+n+t}}{l_x}$. The total probability required must therefore be

$$\begin{aligned} \frac{\Sigma d_{y+n-1}\, l_{x+n+t}}{l_y\, l_x} &= \Sigma_{n+t}p_x({}_{n-1}p_y - {}_np_y) \\ &= {}_tp_x(\Sigma_n p_{x+t} \times {}_{n-1}p_y - \Sigma_n p_{x+t} \times {}_np_y) \\ &= {}_tp_x\left(\frac{e_{x+t\,:\,y-1}}{p_{y-1}} - e_{x+t\,:\,y}\right) \\ \text{or } &= {}_tp_x\{p_{x+t}(1 + e_{x+t+1\,:\,y}) - e_{x+t\,:\,y}\} \end{aligned}$$

7. We may find an alternative formula for $e_{y|x}$ as follows:—

Taking the nth year, the probability of (y) dying therein is $({}_{n-1}p_y - {}_np_y)$, and (x)'s expectation of living to the end of that year and of each year after amounts to

$$({}_np_x + {}_{n+1}p_x + {}_{n+2}p_x + \cdot\ \cdot\ \cdot\ ) = {}_np_x(1 + e_{x+n}).$$

The expectation of (x) after (y), should (y) die in the nth year, is therefore ${}_np_x({}_{n-1}p_y - {}_np_y)(1 + e_{x+n})$, and the total expectation

$$e_{y|x} = \Sigma_n p_x({}_{n-1}p_y - {}_np_y)(1 + e_{x+n}).$$

That this expression is identical with that in the formula

$$e_{y|x} = \Sigma_n p_x(1 - {}_np_y)$$

may be shown as follows :—

$$\begin{aligned}
\Sigma_n p_x(1 - {}_np_y) &= {}_1p_x(1 - {}_1p_y) + {}_2p_x(1 - {}_2p_y) + {}_3p_x(1 - {}_3p_y) + \cdots \\
&= {}_1p_x q_y + {}_2p_x(q_y + {}_{1|}q_y) + {}_3p_x(q_y + {}_{1|}q_y + {}_{2|}q_y) + \cdots \\
&= q_y({}_1p_x + {}_2p_x + {}_3p_x + \cdots) \\
&\qquad + {}_{1|}q_y({}_2p_x + {}_3p_x + \cdots) \\
&\qquad\qquad + {}_{2|}q_y({}_3p_x + \cdots) + \cdots \\
&= q_y p_x(1 + e_{x+1}) + {}_{1|}q_y \times {}_2p_x(1 + e_{x+2}) \\
&\qquad\qquad + {}_{2|}q_y \times {}_3p_x(1 + e_{x+3}) + \cdots \\
&= \Sigma_n p_x({}_{n-1}p_y - {}_np_y)(1 + e_{x+n}).
\end{aligned}$$

8. In finding $e_{x:\overline{y(\overline{t|})}}$ we have, as shown in the *Text Book*, to divide the expectation. Then the expectation for t years depends on (x) alone, and is equal to ${}_{|t}e_x$. Thereafter, in order that (x)'s living to the end of the $(t+1)$th year may count, (y) must live to the end of one year; that (x)'s living to the end of the $(t+2)$th year may count, (y) must live to the end of two years; and so on. We have therefore

$$\begin{aligned}
e_{x:\overline{y(\overline{t|})}} &= {}_{|t}e_x + \frac{l_{x+t+1}\,l_{y+1} + l_{x+t+2}\,l_{y+2} + \cdots}{l_x\,l_y} \\
&= {}_{|t}e_x + {}_tp_x \frac{l_{x+t+1}\,l_{y+1} + l_{x+t+2}\,l_{y+2} + \cdots}{l_{x+t}\,l_y} \\
&= {}_{|t}e_x + {}_tp_x e_{x+t:y} \\
&= e_x - {}_tp_x(e_{x+t} - e_{x+t:y})
\end{aligned}$$

9. Coming to the probability that (x) will die first of the three

lives (x), (y), and (z), we may proceed in the same manner as in the case of two lives.

$$Q^1_{xyz} = \Sigma \frac{d_{x+n-1}}{l_x} \times \frac{l_{y+n-\frac{1}{2}}}{l_y} \times \frac{l_{z+n-\frac{1}{2}}}{l_z}$$

$$= \Sigma \frac{l_{x+n-1} - l_{x+n}}{l_x} \times \frac{l_{y+n-1} + l_{y+n}}{2l_y} \times \frac{l_{z+n-1} + l_{z+n}}{2l_z}$$

$$= \Sigma \frac{1}{4}\left(\frac{l_{x+n-1}\, l_{y+n-1}\, l_{z+n-1}}{l_x l_y l_z} - \frac{l_{x+n}\, l_{y+n-1}\, l_{z+n-1}}{l_x l_y l_z} \right.$$

$$+ \frac{l_{x+n-1}\, l_{y+n-1}\, l_{z+n}}{l_x l_y l_z} - \frac{l_{x+n}\, l_{y+n-1}\, l_{z+n}}{l_x l_y l_z}$$

$$+ \frac{l_{x+n-1}\, l_{y+n}\, l_{z+n-1}}{l_x l_y l_z} - \frac{l_{x+n}\, l_{y+n}\, l_{z+n-1}}{l_x l_y l_z}$$

$$\left. + \frac{l_{x+n-1}\, l_{y+n}\, l_{z+n}}{l_x l_y l_z} - \frac{l_{x+n}\, l_{y+n}\, l_{z+n}}{l_x l_y l_z} \right)$$

$$= \frac{1}{4}\left(1 - \frac{e_{x:y-1:z-1}}{p_{y-1:z-1}} + \frac{e_{x-1:y-1:z}}{p_{x-1:y-1}} - \frac{e_{x:y-1:z}}{p_{y-1}} + \frac{e_{x-1:y:z-1}}{p_{x-1:z-1}} \right.$$

$$\left. - \frac{e_{x:y:z-1}}{p_{z-1}} + \frac{e_{x-1:y:z}}{p_{x-1}} \right)$$

since $\Sigma \left(\frac{l_{x+n-1}\, l_{y+n-1}\, l_{z+n-1}}{l_x l_y l_z} - \frac{l_{x+n}\, l_{y+n}\, l_{z+n}}{l_x l_y l_z} \right) = 1$

The formula given in the *Text Book* differs from the above, exceeding it by

$$\frac{1}{12}\left(1 - \frac{e_{x:y-1:z-1}}{p_{y-1:z-1}} - \frac{e_{x-1:y-1:z}}{p_{x-1:y-1}} - \frac{e_{x-1:y:z-1}}{p_{x-1:z-1}} \right.$$

$$\left. + \frac{e_{x:y-1:z}}{p_{y-1}} + \frac{e_{x:y:z-1}}{p_{z-1}} + \frac{e_{x-1:y:z}}{p_{x-1}} \right)$$

which, however, is a very small quantity.

10. To adapt the expression given above for use with select tables, we have

$$Q^1_{xyz} = \frac{1}{4}\Big\{1 - p_x(1+e_{x+1:y:z}) + p_y(1+e_{x:y+1:z}) + p_z(1+e_{x:y:z+1})$$
$$- p_{xy}(1+e_{x+1:y+1:z}) - p_{xz}(1+e_{x+1:y:z+1}) + p_{yz}(1+e_{x:y+1:z+1})\Big\}$$

since $e_{x:y-1:z-1} = p_{x:y-1:z-1}(1+e_{x+1:y:z})$, etc.

11. A simpler solution of the problem to find the value of $e^1_{yz}|_x$ than that given in the *Text Book* may be suggested. If, in the equation

$$e^1_{yz}|_x = \Sigma_n p_x \times |_n q^1_{yz}$$

it be assumed that, for all values of n, $|_n q^1_{yz} = Q^1_{yz} \times |_n q_{yz}$ (see Articles 7 and 8 *Text Book*, Chap. XV.), we have at once

$$\begin{aligned} e^1_{yz}|_x &= \Sigma_n p_x Q^1_{yz} \times |_n q_{yz} \\ &= Q^1_{yz} e_{yz}|_x \\ &= Q^1_{yz}(e_x - e_{xyz}) \end{aligned}$$

Also

$$\begin{aligned} e^2_{yz}|_x &= e^1_{yz}|_x - e_y|_{xz} \\ &= Q^1_{yz}(e_x - e_{xyz}) - (e_{xz} - e_{xyz}) \\ &= (1 - Q^1_{yz})(e_x - e_{xyz}) - (e_{xz} - e_{xyz}) \\ &= (e_x - e_{xz}) - Q^1_{yz}(e_x - e_{xyz}) \end{aligned}$$

Or we may proceed thus:—

$$\begin{aligned} e^2_{yz}|_x &= \Sigma_n p_x \times |_n q^2_{yz} \\ &= \Sigma_n p_x(|_n q_z - |_n q^1_{yz}) \\ &= e_z|_x - e^1_{yz}|_x \\ &= (e_x - e_{xz}) - Q^1_{yz}(e_x - e_{xyz}) \end{aligned}$$

12. The following should be carefully noted.

There are two formulas for $q_{\overline{xy}}$:—

$$q_{\overline{xy}} = q_x + q_y - q_{xy}$$
$$\text{and} \quad q_{\overline{xy}} = q_x \times q_y$$

We may express in similar forms the probability that two

children, ten and fifteen years old respectively, will both die before attaining age twenty-one, viz. :—

$$|_{11}q_{10} + |_{6}q_{15} - \left(|_{6}q_{10:15} + {}_{6}p_{10:15} \times |_{5}q_{16}\right)$$

and $$|_{11}q_{10} \times |_{6}q_{15}$$

The probability that one at least will so die is

$$1 - (1 - |_{11}q_{10})(1 - |_{6}q_{15}).$$

Now to obtain the probability that both the children will die before twenty-one in the lifetime of their mother aged fifty, we have analogously

$$|_{11}q^{1}_{10:50} + |_{6}q^{1}_{15:50} - \left(|_{6}q^{\;\;1}_{\overline{10:15}:50} + {}_{6}p_{10:15:50} \times |_{5}q^{1}_{16:50}\right)$$

and $$|_{11}q^{1}_{10:50} \times |_{6}q^{1}_{15:50}$$

The probability that one at least will so die is

$$1 - (1 - |_{11}q^{1}_{10:50})(1 - |_{6}q^{1}_{15:50})$$

EXAMPLES

1. Out of l_x married couples, the husbands being all aged x and the wives aged y, show how the number of husbands who become widowers in the nth year may be ascertained.

The number of husbands becoming widowers in the nth year is

$$l_x \times {}_{n-1|}q^{\;1}_{xy} = l_x \times \tfrac{1}{2}({}_{n-1}p_y - {}_{n}p_y)({}_{n-1}p_x + {}_{n}p_x)$$

$$= l_x \times \frac{d_{y+n-1}}{l_y} \times \frac{l_{x+n-\frac{1}{2}}}{l_x}$$

We do not cancel l_x in the numerator with l_x in the denominator, since the former is taken from actual experience, while the latter is taken from a table which represents suitably the mortality of the lives concerned.

2. If the probability that (x) will die before (z) is ·1996;
,, (x) ,, both (y) and (z) is ·1610;
,, (y) ,, (z) is ·2990;
,, (y) ,, both (x) and (z) is ·2602;

find the values of the following probabilities :—

(1) That the survivor of (x) and (y) will die before (z).

(2) That (x) will die before (z), (y) having died first.

$$(1)\quad Q^{\ 1}_{\overline{xy}:z} = Q^{2}_{\underset{1}{xyz}} + Q^{\ 2}_{\underset{1}{xyz}} = Q^{1}_{xz} - Q^{1}_{xyz} + Q^{1}_{yz} - Q^{\ 1}_{xyz}$$

$$= \cdot1996 - \cdot1610 + \cdot2990 - \cdot2602$$

$$= \cdot0386 + \cdot0388$$

$$= \cdot0774$$

$$(2)\quad Q^{2}_{\underset{1}{xyz}} = \cdot0386$$

3. Find the probabilities that, in the tth year from the present time,

(*a*) A life now aged x will die, having survived a life now aged y by at least m years, and a life aged z by at least n years.

(*b*) The last survivor of three lives (x), (y), and (z) will die.

(*c*) A life (z) will die, leaving (x) and (y) surviving.

(*a*) To fulfil the conditions, (x) must die in the tth year, (y) on the average before the middle of the $(t-m)$th year, and (z) on the average before the middle of the $(t-n)$th year respectively. The probability is

$$\frac{d_{x+t-1}}{l_x} \times \frac{l_y - l_{y+t-m-\frac{1}{2}}}{l_y} \times \frac{l_z - l_{z+t-n-\frac{1}{2}}}{l_z}$$

(*b*) $_{t-1|}q_{\overline{xyz}} = {}_{t-1}p_{\overline{xyz}} - {}_tp_{\overline{xyz}}$

$$= ({}_{t-1}p_x + {}_{t-1}p_y + {}_{t-1}p_z - {}_{t-1}p_{xy} - {}_{t-1}p_{xz} - {}_{t-1}p_{yz} + {}_{t-1}p_{xyz})$$

$$- ({}_tp_x + {}_tp_y + {}_tp_z - {}_tp_{xy} - {}_tp_{xz} - {}_tp_{yz} + {}_tp_{xyz})$$

$$= {}_{t-1|}q_x + {}_{t-1|}q_y + {}_{t-1|}q_z - {}_{t-1|}q_{xy} - {}_{t-1|}q_{xz} - {}_{t-1|}q_{yz}$$

$$+ {}_{t-1|}q_{xyz}$$

$$(c)\qquad \frac{d_{z+t-1}}{l_z} \times \frac{l_{x+t-\frac{1}{2}}}{l_x} \times \frac{l_{y+t-\frac{1}{2}}}{l_y}$$

4. Find expressions for the probabilities that of three lives (x) (y) and (z), (x) will die

(1) In the same year as (z), whether first, second, or third of the lives.

(2) At least t years after (y), and at least t years before (z).

(1) This probability is equal to the sum for all values of n of the probability that both (x) and (z) will die in the nth year

$$\begin{aligned}
&= \Sigma({}_{n-1}p_x - {}_np_x)({}_{n-1}p_z - {}_np_z) \\
&= \Sigma({}_{n-1}p_{xz} + {}_np_{xz} - {}_{n-1}p_x \times {}_np_z - {}_np_x \times {}_{n-1}p_z) \\
&= 1 + 2e_{xz} - \frac{e_{x-1:z}}{p_{x-1}} - \frac{e_{x:z-1}}{p_{z-1}}
\end{aligned}$$

(2) If (x) die in the $(n+t+1)$th year, then to fulfil the conditions (y) must on the average have died before the middle of the $(n+1)$th year, and (z) must live on the average till the middle of the $(n+2t+1)$th year. The probability required is therefore

$$\begin{aligned}
&\Sigma \frac{d_{x+n+t}}{l_x} \times \frac{l_y - l_{y+n+\frac{1}{2}}}{l_y} \times \frac{l_{z+n+2t+\frac{1}{2}}}{l_z} \\
&= {}_tp_x \times {}_{2t}p_z \times \Sigma \frac{d_{x+t+n}}{l_{x+t}} \times \frac{l_y - l_{y+n+\frac{1}{2}}}{l_y} \times \frac{l_{z+2t+n+\frac{1}{2}}}{l_{z+2t}} \\
&= {}_tp_x \times {}_{2t}p_z Q^{2}_{\overline{x+t}:\underset{1}{y}:z+2t}
\end{aligned}$$

5. Required (x)'s expectation of life ten years after the death of a life presently aged y.

From the whole expectation of life of (x), we must deduct the part during the life of (y) and for ten years after. The expectation required is therefore

$$\begin{aligned}
e_x - e_{x:\overline{y(10|)}} &= e_x - \{e_x - {}_{10}p_x(e_{x+10} - e_{x+10:y})\} \\
&= {}_{10}p_x(e_{x+10} - e_{x+10:y})
\end{aligned}$$

Or we may proceed otherwise. The $(n+10)$th year will count only if (x) survive it, and if (y) die within n years. The probability of its being reckoned is therefore $(1 - {}_np_y)_{n+10}p_x$, and the sum of this expression for all values of n is the expectation required

$$\begin{aligned}
&= \Sigma(1 - {}_np_y)_{n+10}p_x \\
&= {}_{10}p_x\Sigma(1 - {}_np_y)_np_{x+10} \\
&= {}_{10}p_x(e_{x+10} - e_{x+10:y})
\end{aligned}$$

6. Write down in respect of the $(t+1)$th year the probability indicated by each of the following symbols:—(a) $Q^2_{x\underset{1}{y}z}$, (b) Q^2_{xyz}, (c) Q^3_{xyz}, (d) $Q^1_{x:\overline{yz}}$, (e) $Q^{\,1}_{\overline{xy}:z}$, (f) $Q^{\,1}_{\overline{'xy'}:z}$

(a) $\dfrac{1}{l_x l_y l_z}\, d_{x+t}(l_y - l_{y+t+\frac{1}{2}})\, l_{z+t+\frac{1}{2}}$

(b) $\dfrac{1}{l_x l_y l_z}\, d_{x+t}\{(l_y - l_{y+t+\frac{1}{2}})l_{z+t+\frac{1}{2}} + (l_z - l_{z+t+\frac{1}{2}})l_{y+t+\frac{1}{2}}\}$

(c) $\dfrac{1}{l_x l_y l_z}\, d_{x+t}(l_y - l_{y+t+\frac{1}{2}})(l_z - l_{z+t+\frac{1}{2}})$

(d) $\dfrac{1}{l_x l_y l_z}\, d_{x+t}\{l_{y+t+\frac{1}{2}}\, l_{z+t+\frac{1}{2}} + (l_y - l_{y+t+\frac{1}{2}})l_{z+t+\frac{1}{2}}$
$+ (l_z - l_{z+t+\frac{1}{2}})\, l_{y+t+\frac{1}{2}}\}$

(e) $\dfrac{1}{l_x l_y l_z}\, l_{z+t+\frac{1}{2}}\{(l_x - l_{x+t+\frac{1}{2}})d_{y+t} + (l_y - l_{y+t+\frac{1}{2}})d_{x+t}\}$

(f) $\dfrac{1}{l_x l_y l_z}\, l_{z+t+\frac{1}{2}}(l_{x+t+\frac{1}{2}}\, d_{y+t} + l_{y+t+\frac{1}{2}}\, d_{x+t})$

7. Show that $Q^1_{xy} = \dfrac{1}{2}\left(\dfrac{\mathring{e}_{x-1:y}}{p_{x-1}} - p_x \mathring{e}_{x+1:y}\right)$ approximately.

$$Q^1_{xy} = \frac{1}{l_x l_y}\int_0^\infty l_{x+t}\mu_{x+t}\, l_{y+t}\,dt$$
$$= \frac{1}{l_x l_y}\int_0^\infty l_{x+t}\, l_{y+t}\,\frac{l_{x+t-1} - l_{x+t+1}}{2l_{x+t}}\,dt \text{ approximately.}$$
$$= \frac{1}{2l_x l_y}\left(\int_0^\infty l_{x+t-1}\, l_{y+t}\,dt - \int_0^\infty l_{x+t+1}\, l_{y+t}\,dt\right)$$
$$= \frac{1}{2}\left(\frac{1}{p_{x-1}}\int_0^\infty \frac{l_{x+t-1}\, l_{y+t}}{l_{x-1}\, l_y}\,dt - p_x\int_0^\infty \frac{l_{x+t+1}\, l_{y+t}}{l_{x+1}\, l_y}\,dt\right)$$
$$= \frac{1}{2}\left(\frac{\mathring{e}_{x-1:y}}{p_{x-1}} - p_x \mathring{e}_{x+1:y}\right)$$

since $\mathring{e} = \displaystyle\int_0^\infty \frac{l_{+t}}{l}\,dt$

CHAPTER V

Statistical Applications of the Mortality Table

THE intricate problems relating to population in this chapter may be simplified by the following explanations.

1. In *Text Book,* Article 14, we are told "the average age at death of a stationary population, kept up by births alone, is $\frac{T_0}{l_0} = \mathring{e}_0$," which is explained as follows:—

The d_0 persons who die between 0 and 1 are on the average $\frac{1}{2}$ year old at death, that is, amongst them they have lived $\frac{1}{2}d_0$ years.

The d_1 persons who die between 1 and 2 are on the average $1\frac{1}{2}$ years, etc.

The d_2 persons who die between 2 and 3 are on the average $2\frac{1}{2}$ years, etc.

Therefore, the $d_0 + d_1 + d_2 +$ etc. persons who die have, taken altogether, lived $\frac{1}{2}d_0 + \frac{3}{2}d_1 + \frac{5}{2}d_2 +$ etc. years, and the average age at death of each is therefore

$$\frac{\frac{1}{2}d_0 + \frac{3}{2}d_1 + \frac{5}{2}d_2 + \text{ etc.}}{d_0 + d_1 + d_2 + \text{ etc.}} = \frac{L_0 + L_1 + L_2 + \text{ etc.}}{l_0}$$
$$= \frac{T_0}{l_0}$$
$$= \mathring{e}_0$$

To find the average age at death of those who are aged x and upwards, we have by a similar process such average age

$$= \frac{(x+\frac{1}{2})d_x + (x+\frac{3}{2})d_{x+1} + (x+\frac{5}{2})d_{x+2} + \cdot\ \cdot\ \cdot}{d_x + d_{x+1} + d_{x+2} + \cdot\ \cdot\ \cdot}$$
$$= x + \frac{L_x + L_{x+1} + L_{x+2} + \cdot\ \cdot\ \cdot}{l_x}$$
$$= x + \frac{T_x}{l_x}$$
$$= x + \mathring{e}_x$$

Again, similarly, the average age at death of those who die within the n years after age x is

$$\frac{(x+\frac{1}{2})d_x+(x+\frac{3}{2})d_{x+1}+\cdot\cdot\cdot+(x+n-\frac{1}{2})d_{x+n-1}}{d_x+d_{x+1}+\cdot\cdot\cdot+d_{x+n-1}}$$

$$= x+\frac{\frac{1}{2}d_x+\frac{3}{2}d_{x+1}+\cdot\cdot\cdot+(n-\frac{1}{2})d_{x+n-1}}{l_x-l_{x+n}}$$

$$= x+\frac{\mathrm{T}_x-\mathrm{T}_{x+n}-nl_{x+n}}{l_x-l_{x+n}}$$

since $\frac{1}{2}d_x+\frac{3}{2}d_{x+1}+\cdot\cdot\cdot+(n-\frac{1}{2})d_{x+n-1}$

$$= (\tfrac{1}{2}d_x+\tfrac{3}{2}d_{x+1}+\tfrac{5}{2}d_{x+2}+\cdot\cdot\cdot)$$
$$-(\tfrac{1}{2}d_{x+n}+\tfrac{3}{2}d_{x+n+1}+\tfrac{5}{2}d_{x+n+2}+\cdot\cdot\cdot)$$
$$-n(d_{x+n}+d_{x+n+1}+d_{x+n+2}+\cdot\cdot\cdot)$$
$$= \mathrm{T}_x-\mathrm{T}_{x+n}-nl_{x+n}$$

We are now in a position to state the solution to the problem in *Text Book*, Article 14, as to the effect at the end of n years, on the average age at death of an otherwise stationary population, disturbed by "an annual influx of $\frac{1}{\kappa}\times l_x$ persons aged x."

The total years lived by those that have died out of the original population are $(\frac{1}{2}d_0+\frac{3}{2}d_1+\frac{5}{2}d_2+\cdot\cdot\cdot)$, and the total years at death lived by the immigrants are

$$\frac{1}{\kappa}\left\{(x+\tfrac{1}{2})d_x+(x+\tfrac{3}{2})d_{x+1}+\cdot\cdot\cdot+(x+n-\tfrac{1}{2})d_{x+n-1}\right\}$$

Also the whole number of deaths of both is

$$(d_0+d_1+d_2+\cdot\cdot\cdot)+\frac{1}{\kappa}(d_x+d_{x+1}+\cdot\cdot\cdot+d_{x+n-1})$$

and therefore the average age at death of each who dies is

$$\frac{(\frac{1}{2}d_0+\frac{3}{2}d_1+\frac{5}{2}d_2+\cdots)+\frac{1}{\kappa}\left\{(x+\frac{1}{2})d_x+(x+\frac{3}{2})d_{x+1}+\cdots+(x+n-\frac{1}{2})d_{x+n-1}\right\}}{(d_0+d_1+d_2+\cdot\cdot\cdot)+\frac{1}{\kappa}(d_x+d_{x+1}+\cdot\cdot\cdot+d_{x+n-1})}$$

$$= \frac{\mathrm{T}_0+\frac{1}{\kappa}\left\{(l_x-l_{x+n})x+(\mathrm{T}_x-\mathrm{T}_{x+n}-nl_{x+n})\right\}}{l_0+\frac{1}{\kappa}(l_x-l_{x+n})}$$

2. Passing to the problems discussed in *Text Book,* Articles 18 to 20, we may find the average present age of the existing population as follows:—

There are L_0 persons living between 0 and 1, and they have lived on the average $\frac{1}{2}$ year each.

There are L_1 persons living between 1 and 2, and they have lived on the average $1\frac{1}{2}$ years each.

There are L_2 persons living between 2 and 3, and they have lived on the average $2\frac{1}{2}$ years each, etc.

Therefore the total years lived amount to $(\frac{1}{2}L_0 + \frac{3}{2}L_1 + \frac{5}{2}L_2 +$ etc.) and the average lived by each person is

$$\frac{\frac{1}{2}L_0 + \frac{3}{2}L_1 + \frac{5}{2}L_2 + \cdots}{L_0 + L_1 + L_2 + \cdots} = \frac{\frac{1}{2}T_0 + T_1 + T_2 + \cdots}{T_0}$$

$$= \frac{Y_0}{T_0}$$

Similarly the average present age of the population who are aged x and upwards is

$$\frac{(x+\frac{1}{2})L_x + (x+\frac{3}{2})L_{x+1} + (x+\frac{5}{2})L_{x+2} + \cdots}{L_x + L_{x+1} + L_{x+2} + \cdots}$$

$$= x + \frac{\frac{1}{2}T_x + T_{x+1} + T_{x+2} + \cdots}{T_x}$$

$$= x + \frac{Y_x}{T_x}$$

3. Again, to find the average future lifetime of the existing population, we must first note that the average future lifetime of l_x persons is T_x. Hence the average future lifetime of L_x persons, *i.e.*, $\frac{1}{2}(l_x + l_{x+1})$ persons, is $\frac{1}{2}(T_x + T_{x+1})$, and the average future lifetime of each of these L_x persons is therefore

$$\frac{\frac{1}{2}(T_x + T_{x+1})}{L_x} \text{ years.}$$

Now the average future lifetime of L_0 persons living between

0	and	1	will be	$\frac{1}{2}(T_0+T_1)$,	and of	L_1	„
1	„	2	„	$\frac{1}{2}(T_1+T_2)$,	and of	L_2	„
2	„	3	„	$\frac{1}{2}(T_2+T_3)$,	and so on.		

Therefore the average future lifetime of each of the whole existing population will be

$$\frac{\frac{1}{2}(T_0+T_1)+\frac{1}{2}(T_1+T_2)+\frac{1}{2}(T_2+T_3)+\cdots}{L_0+L_1+L_2+\cdots}$$

$$=\frac{\frac{1}{2}T_0+T_1+T_2+\cdots}{T_0}$$

$$=\frac{Y_0}{T_0}$$

Similarly the average future lifetime of the existing population who are aged x and upwards may be shown to be $\frac{Y_x}{T_x}$.

4. Now if the average present age of the existing population is $\frac{Y_0}{T_0}$, and if their average future lifetime is $\frac{Y_0}{T_0}$, it is obvious that the average age at death of the existing population will be $\frac{2Y_0}{T_0}$; and for those of the existing population who are aged x and upwards the average age at death will be $x+\frac{2Y_x}{T_x}$.

5 The distinction drawn in *Text Book,* Article 21, may be shown thus:—

In the case of the stationary population which is recruited each year by births, we have for the average age at death

$$\frac{\frac{1}{2}d_0+\frac{3}{2}d_1+\frac{5}{2}d_2+\cdots}{d_0+d_1+d_2+\cdots}$$

$$=\frac{T_0}{l_0}$$

$$=\mathring{e}_0$$

In the case of the existing population we have as follows:—

Age. (1)	Number living. (2)	Average age at death of each. (3)
0-1	L_0	$\frac{1}{2}+\frac{\frac{1}{2}(T_0+T_1)}{L_0}$
1-2	L_1	$\frac{3}{2}+\frac{\frac{1}{2}(T_1+T_2)}{L_1}$
2-3	L_2	$\frac{5}{2}+\frac{\frac{1}{2}(T_2+T_3)}{L_2}$
etc.	etc.	etc.

Multiplying the number living at each age by the average age at death of each of the group, and dividing the sum of these products by the total population, we have for the average age at death of the existing population

$$\frac{\frac{1}{2}L_0+\frac{3}{2}L_1+\frac{5}{2}L_2+\cdots}{L_0+L_1+L_2+\cdots}+\frac{\frac{1}{2}(T_0+T_1)+\frac{1}{2}(T_1+T_2)+\frac{1}{2}(T_2+T_3)+\cdots}{L_0+L_1+L_2+\cdots}$$

$$=\frac{\frac{1}{2}T_0+T_1+T_2+\cdots}{T_0}+\frac{\frac{1}{2}T_0+T_1+T_2+\cdots}{T_0}$$

$$=\frac{Y_0}{T_0}+\frac{Y_0}{T_0}$$

$$=\frac{2Y_0}{T_0}$$

In the case of $\mathring{e}_0$ we have the average age at death of the population and, assuming that there are l_0 annual births, this average age is the same every year. In the case of $\frac{2Y_0}{T_0}$ we multiply the number at present living in each age group by the average age at death of the group, and by this process obtain the average age at death of the present members of the community.

Applying the same process to those of the present population aged x and upwards we have for their average age at death

$$\frac{(x+\frac{1}{2})L_x+(x+\frac{3}{2})L_{x+1}+(x+\frac{5}{2})L_{x+2}+\cdots}{L_x+L_{x+1}+L_{x+2}+\cdots}+\frac{\frac{1}{2}T_x+T_{x+1}+T_{x+2}+\cdots}{L_x+L_{x+1}+L_{x+2}+\cdots}$$

$$=x+\frac{Y_x}{T_x}+\frac{Y_x}{T_x}$$

$$=x+\frac{2Y_x}{T_x}$$

EXAMPLES

1. Having given a complete table of p_x, accurately representing the probabilities of life at all ages, show how, from the deaths taking place in one year, to calculate approximately the total numbers living in a stationary population where there is no disturbance from immigration or emigration.

Since the population is stationary, the deaths in any one year must equal the births, that is

$$l_0 = d_0 + d_1 + d_2 + d_3 + \cdot \cdot \cdot$$

We are also given p_0, p_1, p_2, p_3, etc., and therefore since

$$\begin{aligned} l_1 &= l_0 \times p_0 \\ l_2 &= l_1 \times p_1 \\ l_3 &= l_2 \times p_2 \\ &\text{etc.} \end{aligned}$$

we can successively obtain l_1, l_2, l_3, etc.

Now making the assumption of a uniform distribution of births and deaths, the total population numbers

$$\begin{aligned} &\tfrac{1}{2}(l_0 + l_1) + \tfrac{1}{2}(l_1 + l_2) + \tfrac{1}{2}(l_2 + l_3) + \cdot \cdot \cdot \\ &= \tfrac{1}{2}l_0 + l_1 + l_2 + l_3 + \cdot \cdot \cdot \end{aligned}$$

2. A military power desires to maintain a standing army of 1,000,000 men. Five years' service is compulsory on all males attaining the age of twenty. How would you apply a table, showing the mortality amongst males, to ascertain the annual number of recruits required to maintain the army at its proper strength?

By the table, an entry at age twenty of l_{20} males will support a population between twenty and twenty-five of $T_{20} - T_{25}$. By simple proportion we find the number of recruits of age twenty necessary to support an army of 1,000,000 men of ages twenty to twenty-five to be $\frac{1000000}{T_{20} - T_{25}} l_{20}$.

This formula naturally only takes account of the numbers in time of peace. It further does not allow for the necessary selection by medical examination of the recruits, nor for the effect thereof on mortality.

3. In a stationary community supported by 5000 annual births, each member, on attaining the age of twenty, makes a payment of £20, and contributes £1 at the end of each succeeding year until, and inclusive of, the sixtieth birthday; receiving thereafter an annuity of £15, payable at the end of each year. In respect of each contributor who dies before receiving the first payment of £15, a payment of £5 is made. Find expressions for (*a*) the number of contributors, (*b*) the annual receipts, (*c*) the total yearly annuity-payment, and (*d*) the annual death claims.

(*a*) The number of contributors is $\frac{5000}{l_0}(T_{20} - T_{60})$.

(*b*) The annual receipts are

$$\frac{5000}{l_0}(20l_{20} + l_{21} + l_{22} + \cdots + l_{60}) = \frac{5000}{l_0}(20l_{20} + N'_{20} - N'_{60})$$

(*c*) The total yearly annuity-payment is

$$\frac{5000}{l_0} \times 15(l_{61} + l_{62} + l_{63} + \cdot \cdot \cdot + l_{\omega-1}) = \frac{5000}{l_0} \times 15N'_{60}$$

(*d*) The annual death claims are

$$\frac{5000}{l_0} \times 5(d_{20} + d_{21} + d_{22} + \cdot \cdot \cdot + d_{60}) = \frac{5000}{l_0} \times 5(l_{20} - l_{61})$$

4. Explain clearly the difference between $\frac{T_0}{l_0}$ and $\frac{2Y_0}{T_0}$. What does $\frac{T_x - T_{x+n}}{l_x}$ represent?

A community otherwise stationary is subject for n years to an annual increase from immigration at age x to the extent of 10 per cent. of the number who attain that age. Show how to ascertain the effect of this immigration upon the average age at death at the end of n years. What practical consideration would vitiate your result?

$\frac{T_0}{l_0}$ is the average age at death of each of l_0 persons born in a stationary community, while as explained in *Text Book*, Article 21 of this chapter, $\frac{2Y_0}{T_0}$ is the average age at death of the present members of this community.

$\frac{T_x - T_{x+n}}{l_x}$ is the average number of years lived by each of l_x persons between ages x and $(x+n)$. See *Text Book*, Article 12.

To get the average age at death in the above described community, put $\frac{1}{\kappa} = \frac{1}{10}$ in the formula in *Text Book*, Article 14.

The result is vitiated as shown in *Text Book*, Article 16, by reason of no increase in births following on the increase in population being taken into account.

5. In a population of 1,000,000, hitherto stationary, the birth-rate begins to increase at the rate of 1 per cent. per annum. What is the population at the end of three years, assuming a uniform distribution of births and deaths throughout the year?

According to the mortality table, a population of T_0 must be supported by l_0 births, and therefore a population of 1,000,000 must be supported by $l_0 \times \dfrac{1000000}{T_0}$ births.

At the end of a period of three years, the population due to the increase in the birth-rate is

$$l_0 \times \frac{1000000}{T_0}\Big[\{(1{\cdot}01)^3 - 1\}L_0 + \{(1{\cdot}01)^2 - 1\}L_1 + {\cdot}01L_2\Big]$$
$$= l_0 \times \frac{1000000}{T_0}({\cdot}030301L_0 + {\cdot}0201L_1 + {\cdot}01L_2)$$

Hence the total population is

$$1000000 + \frac{l_0}{T_0}(30301L_0 + 20100L_1 + 10000L_2)$$

6. In a certain community the number of annual births has been observed to decrease approximately in a geometrical progression. It is desired to introduce a pension scheme, pensions to commence at age sixty-five, the contribution being from age twenty to age fifty-five. The number of births this year being k, find expressions for the immediate numbers of pensioners and contributors.

Let $\dfrac{1}{r}$ be the common ratio of the geometrical progression in which the annual births are decreasing. Then it follows that the births n years ago were $r^n k$.

Assuming that the births take place uniformly throughout the calendar year, that the figures are required at the close of the calendar year, and that there is no disturbance from immigration or emigration, we have the number of immediate pensioners

$$\frac{k}{l_0}(r^{65}L_{65} + r^{66}L_{66} + r^{67}L_{67} + \cdot\ \cdot\ \cdot\ )$$

and the number of contributors

$$\frac{k}{l_0}(r^{20}L_{20} + r^{21}L_{21} + r^{22}L_{22} + \cdot\ \cdot\ \cdot + r^{54}L_{54}).$$

7. A system of free education is introduced into a community, embracing all children from the age of five to thirteen inclusive. The present number of such children is A, but the birth-rate in the community, previously stationary, has been increasing during the last four years at 1 per cent. per annum. What would you estimate to be the number of children under education at the end of twenty years, and how many children will have passed out during the period?

Assuming that, during the period the population was stationary, a table of mortality was formed showing the column of L and the column of T, we may proceed as follows:—

In the first year of the system the number under education is

$$A\left(\frac{L_5+L_6+L_7+\cdot\cdot\cdot+L_{13}}{T_5-T_{14}}\right)$$

In the second year $A\left\{\frac{L_5(1\cdot01)+L_6+L_7+\cdot\cdot\cdot+L_{13}}{T_5-T_{14}}\right\}$

In the third year $A\left\{\frac{L_5(1\cdot01)^2+L_6(1\cdot01)+L_7+\cdot\cdot\cdot+L_{13}}{T_5-T_{14}}\right\}$

and so on, till at the end of the twentieth year the number will be

$$A\left\{\frac{L_5(1\cdot01)^{19}+L_6(1\cdot01)^{18}+L_7(1\cdot01)^{17}+\cdot\cdot\cdot+L_{13}(1\cdot01)^{11}}{T_5-T_{14}}\right\}$$

The number that will have passed out during the twenty years, having attained the age of fourteen, will be

$$\frac{A\times l_{14}}{T_5-T_{14}}\left\{10+(1\cdot01)+(1\cdot01)^2+\cdot\cdot\cdot+(1\cdot01)^{10}\right\}$$

8. In a pension society it is a condition that if a member dies after m years from entry his widow is entitled to an annuity. If on the average there enter the society in the course of the year k new members aged x, each with a wife aged y, how many widows entitled to draw annuities will there be at the end of the nth year of the society's existence?

The widows between $(y+m)$ and $(y+m+1)$ years of age are on the average each $(y+m+\frac{1}{2})$, and that they may be entitled to annuities their husbands must have died between $(x+m)$ and $(x+m+\frac{1}{2})$. Similarly the husbands of the widows whose average age is $(y+m+\frac{3}{2})$ must have died between $(x+m)$ and $(x+m+\frac{3}{2})$

if their widows are to get annuities, and so on for $(n-m)$ terms. Therefore the number of widows entitled to draw annuities is

$$\frac{k}{l_x l_y}\left\{l_{y+m+\frac{1}{2}}(l_{x+m}-l_{x+m+\frac{1}{2}})+l_{y+m+\frac{3}{2}}(l_{x+m}-l_{x+m+\frac{3}{2}})+\cdots\right.$$
$$\left.+l_{y+n-\frac{1}{2}}(l_{x+m}-l_{x+n-\frac{1}{2}})\right\}$$
$$=k({}_mp_x\times{}_{m|n-m}\mathring{e}_y-{}_{m|n-m}\mathring{e}_{xy})$$

An alternative method of arriving at the same result is to deduct from the number of females who are alive and who have become entitled to annuities by the survivance of their husbands for the necessary m years the number of females who are alive and whose husbands are still alive. The result is obviously the number of widows who are entitled to draw annuities. Following out this plan we get

$$\frac{k}{l_x l_y}\left\{(l_{y+m+\frac{1}{2}}\,l_{x+m}+l_{y+m+\frac{3}{2}}\,l_{x+m}+\cdots+l_{y+n-\frac{1}{2}}\,l_{x+m})\right.$$
$$\left.-(l_{y+m+\frac{1}{2}}\,l_{x+m+\frac{1}{2}}+l_{y+m+\frac{3}{2}}\,l_{x+m+\frac{3}{2}}+\cdots+l_{y+n-\frac{1}{2}}\,l_{x+n-\frac{1}{2}})\right\}$$
$$=k({}_mp_x\times{}_{m|n-m}\mathring{e}_y-{}_{m|n-m}\mathring{e}_{xy})\text{ as above.}$$

9. A railway staff in a stationary condition is recruited annually by 500 entrants at age twenty, who are required to contribute to a pension fund. At age sixty they have the option of retiring on pension, and retirement is compulsory at age sixty-five. Assuming that there are no secessions other than by death, and that one-half of those who reach age sixty retire at that age, the others remaining till age sixty-five, give expressions for:—

(1) The total number of present contributors.
(2) The total number of present pensioners.
(3) The total number of future years' service with which existing contributors will be credited.

(1) The total number of present contributors is

$$\frac{500}{l_{20}}\left\{(L_{20}+L_{21}+\cdots+L_{64})-\tfrac{1}{2}(L_{60}+L_{61}+\cdots+L_{64})\right\}$$
$$=\frac{500}{l_{20}}\left\{(T_{20}-T_{65})-\tfrac{1}{2}(T_{60}-T_{65})\right\}$$
$$=\frac{500}{l_{20}}\left\{T_{20}-\tfrac{1}{2}(T_{60}+T_{65})\right\}$$

(2) The total number of present pensioners is

$$\frac{500}{l_{20}}\left\{\tfrac{1}{2}(L_{60}+L_{61}+\cdot\cdot\cdot+L_{64})+(L_{65}+L_{66}+\cdot\cdot\cdot)\right\}$$

$$=\frac{500}{l_{20}}\left\{\tfrac{1}{2}(T_{60}-T_{65})+T_{65}\right\}$$

$$=\frac{500}{l_{20}}\times\tfrac{1}{2}(T_{60}+T_{65})$$

(The total of present contributors and present pensioners makes up the whole population of age twenty and upwards. This is obviously correct, and goes to prove our results.)

(3) $\frac{1}{2}(T_{20}+T_{21})$ represents the total future lifetime of the L_{20} persons living between twenty and twenty-one. Of this number of years, however, $\frac{1}{2}(T_{60}+T_{65})$ is lived after retirement. Hence $\frac{1}{2}(T_{20}+T_{21})-\frac{1}{2}(T_{60}+T_{65})$ represents the number of future years' service in respect of the L_{20} persons living between twenty and twenty-one.

Similarly $\frac{1}{2}(T_{21}+T_{22})-\frac{1}{2}(T_{60}+T_{65})$ represents the number of future years' service in respect of the L_{21} persons living between twenty-one and twenty-two.

We shall obtain similar expressions for L_{22}, L_{23}, etc., to L_{59}, that for the last being $\frac{1}{2}(T_{59}+T_{60})-\frac{1}{2}(T_{60}+T_{65})$.

Further, for the $\frac{1}{2}L_{60}$ persons between sixty and sixty-one who have not retired the number of future years' service is $\frac{1}{2}\{\frac{1}{2}(T_{60}+T_{61})-T_{65}\}$. Also for the $\frac{1}{2}L_{61}$ persons between sixty-one and sixty-two years of age the expression is $\frac{1}{2}\{\frac{1}{2}(T_{61}+T_{62})-T_{65}\}$; and so on, till finally for $\frac{1}{2}L_{64}$ it is $\frac{1}{2}\{\frac{1}{2}(T_{64}+T_{65})-T_{65}\}$.

The total number of future years' service with which existing contributors will be credited is therefore

$$\frac{500}{l_{20}}\left\{(Y_{20}-Y_{60})-20(T_{60}+T_{65})+\tfrac{1}{2}(Y_{60}-Y_{65}-5T_{65})\right\}.$$

CHAPTER VI

Formulas of De Moivre, Gompertz, and Makeham, for the Law of Mortality

1. Under the supposition named at the beginning of *Text Book*, Article 7, viz., that the numbers living at successive ages are in geometrical progression, it may be shown as follows that "there would be no assignable limit to the duration of human life, and the values of annuities would be equal at all ages."

Let κ be the radix of the table, and r the rate at which the population is decreasing, and therefore fractional. Then the numbers living at successive ages will be κ, κr, κr^2, κr^3, etc., to infinity.

At any age x we have the force of mortality

$$\begin{aligned}\mu_x &= -\frac{1}{l_x}\frac{dl_x}{dx}\\ &= -\frac{1}{\kappa r^x}\frac{d\kappa r^x}{dx}\\ &= -\frac{1}{\kappa r^x}\kappa r^x \log_e r\\ &= -\log_e r\end{aligned}$$

which is independent of the value of x, and therefore constant at all ages.

Also the complete expectation of life at any age x

$$\begin{aligned}\mathring{e}_x &= \frac{1}{l_x}\int_0^\infty l_{x+t}\,dt\\ &= \frac{1}{\kappa r^x}\int_0^\infty \kappa r^{x+t}\,dt\\ &= \int_0^\infty r^t\,dt\\ &= -\frac{1}{\log_e r}\\ &= \frac{1}{\mu_x}\end{aligned}$$

$\mathring{e}_x$ is therefore also constant at all ages; and thus the first part of the statement is proved.

Anticipating by a little the theory of annuities, we have for the annuity value under the supposition

$$
\begin{aligned}
\bar{a}_x &= \frac{1}{l_x}\int_0^\infty v^t l_{x+t}\,dt \\
&= \frac{1}{\kappa r^x}\int_0^\infty v^t \kappa r^{x+t}\,dt \\
&= \int_0^\infty v^t\, r^t dt \\
&= -\frac{1}{\log_e v + \log_e r} \\
&= \frac{1}{\delta + \mu_x}
\end{aligned}
$$

which as above is independent of x, and is therefore constant at all ages. Thus we find the second part of the statement correct.

2. Under Gompertz's law the formula for l_x is expressed in terms of three constants. The constant k does not vary with the age, and is therefore of importance only in fixing the radix of the mortality table.

From any three values of the number living, the three constants may be deduced. For example, taking the values of log l at ages x, $(x+t)$, and $(x+2t)$, we proceed as follows:—

$$
\begin{aligned}
\log l_x &= \log k + c^x \log g \\
\log l_{x+t} &= \log k + c^{x+t} \log g \\
\log l_{x+2t} &= \log k + c^{x+2t} \log g
\end{aligned}
$$

Taking the first differences of both sides of the equation

$$
\begin{aligned}
\log {}_tp_x &= c^x(c^t - 1)\log g \\
\log {}_tp_{x+t} &= c^{x+t}(c^t - 1)\log g.
\end{aligned}
$$

Now dividing each side of the second equation by the corresponding side of the first, and taking the logarithm of the result, we have

$$\log(\log {}_tp_{x+t}) - \log(\log {}_tp_x) = t \log c,$$

from which c may be found, and by substituting its value in one or other of the second set of equations, $\log g$ may also be found. Again by substituting these two values in the first equation of all,

log k may be found, and from the three constants now known the table may be wholly determined.

3. In the case of a mortality table following Gompertz's law it may be shown that

$$|_t q^1_{xy} = Q^1_{xy} \times |_t q_{xy}$$

a relation which, it was suggested in the notes on Chapter IV., might be used as an approximation in any mortality table.

$$Q^1_{xy} = \frac{1}{l_x l_y}\int_0^\infty l_{x+t}\, l_{y+t}\,\mu_{x+t}\,dt$$

By Gompertz's law

$$= \frac{1}{l_x l_y}\int_0^\infty l_{x+t}\, l_{y+t}\,\mathrm{B}c^{x+t}dt$$

$$= \frac{c^x}{c^x+c^y}\frac{1}{l_x l_y}\int_0^\infty l_{x+t}\, l_{y+t}\,\mathrm{B}c^t(c^x+c^y)dt$$

$$= \frac{c^x}{c^x+c^y}\frac{1}{l_x l_y}\int_0^\infty l_{x+t}\, l_{y+t}(\mu_{x+t}+\mu_{y+t})dt$$

$$= \frac{c^x}{c^x+c^y}\,Q_{xy}$$

$$= \frac{c^x}{c^x+c^y}$$

Also

$$|_t q^1_{xy} = \frac{1}{l_x l_y}\int_0^t l_{x+t}\, l_{y+t}\,\mu_{x+t}\,dt$$

$$= \frac{c^x}{c^x+c^y}\frac{1}{l_x l_y}\int_0^t l_{x+t}\, l_{y+t}(\mu_{x+t}+\mu_{y+t})dt$$

$$= Q^1_{xy} \times |_t q_{xy}$$

4. From consideration of the actual figures in the Carlisle, Seventeen Offices, and Government Annuitants' Tables, Makeham deduced the principle of his first modification of Gompertz's law, namely, that "the probabilities of living, increased or diminished by a certain constant ratio, form a series whose logs are in geometrical progression." If, then, Gompertz's law is defined in the statement that the logarithms of the probabilities of living are in geometrical progression, Makeham's modification thereof is covered by the statement that the first differences of the said logarithms are in geometrical progression with common ratio c, which is shown to be true by the process followed in *Text Book*, Article 19.

It may be noted in passing that Mr King is perhaps not

strictly correct in suggesting that Gompertz had foreseen Makeham's modification, but had not arrived at formulas to express it, for, as Makeham points out, he would in that case have expressed himself thus :—"It is possible that death may be the consequence *of one or other* of two generally coexisting causes," and so on.

Examination of mortality tables based on the experience of assured lives convinced Makeham that while his modification of the law accounted for the facts very closely from about age thirty-two onwards, the agreement was not so close at the early ages, and he proposed to have recourse to a complementary series during this portion of the mortality table. The want of agreement with the observed facts is due to the fact that the great proportion of the lives under observation at these early ages have recently been medically examined; in other words, it is due to the unequal effect of selection.

The following table gives the rates of mortality of the H^{MF} experience (1) as graduated by Makeham's law, and (2) as adjusted without reference to any formula. It will be noticed that at this portion of the table Makeham's law considerably over-estimates the rates of mortality.

Age.	Assuming Makeham's Formula.	Assuming no Formula.
10	·00679	·00442
11	·00682	·00409
12	·00684	·00388
13	·00686	·00381
14	·00688	·00385
15	·00693	·00404
16	·00695	·00436
17	·00700	·00483
18	·00704	·00543
19	·00709	·00604
20	·00714	·00649
21	·00718	·00679
22	·00725	·00692
23	·00732	·00695
24	·00739	·00695
25	·00748	·00700
26	·00755	·00710
27	·00766	·00733
28	·00775	·00759
29	·00787	·00783
30	·00800	·00806

The same difficulty presents itself in applying the formula to the graduation of annuity statistics, where the mortality is disturbed for too long a period of the table by the introduction of new entrants.

The French experience, both of assured lives and annuitants, was graduated by Makeham's law. The formula was not, however, applied below age twenty-five, the rates of mortality at these ages (which were made identical in the two experiences) being obtained by assuming that q_x could be represented by the following algebraical expression:—

$$q_x = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6,$$

A, B, C, etc., being obtained from seven equations depending on the unadjusted data.

One curious point connected with Makeham's law is the fact that the value of $\log c$ as deduced from different mortality experiences is very nearly identical. The following examples may be given:—

Seventeen Offices	·03956
H^{MF}	·0400008
Thirty American Offices . . .	·041280
Gotha Life Office	·039625
$O^{[M]}$	·039

Commenting on this, Makeham has said: "The practically identical agreement in the value of log c in these several instances could only result from the rate of deterioration of the vital force being the same for each individual. Thus extended, Gompertz's law may be stated as follows: The vital force or recuperative power of each individual loses equal proportions in equal times; and the proportion of vital force so lost by each is universally the same, being approximately represented by $\log c = {\cdot}04$."

Makeham's first modification of Gompertz's law was never intended to be applied to every mortality table. In some cases it was shown to fail to represent the experience. Where geometrical second differences of $\log l_x$ were not found to exist, he suggested that geometrical third differences might exist, "not as in any way superseding the method of second geometrical differences, but merely as a substitute which may be available in certain cases where the other is found to be unsuitable." This

method lacks the neatness of application to joint-life contingencies possessed by the well-known first modification, and has never proved to be of more than theoretical interest. It may simply be stated that under the second modification l_x is of the form $ks^x g^{c^x} w^{x^2}$, whence we find that $\mu_x = A+Hx+Bc^x$.

EXAMPLES

1. Find, on De Moivre's hypothesis, the most probable number of deaths among 1000 persons aged thirty.

The most probable number of deaths may be found by determining the greatest term in the series $(p_{30}+q_{30})^{1000}$.

But by De Moivre's hypothesis $p_{30} = \frac{l_{31}}{l_{30}} = \frac{86-31}{86-30} = \frac{55}{56}$

and $q_{30} = \frac{1}{56}$.

Therefore we require the greatest term in the expansion of $\left(\frac{55}{56}+\frac{1}{56}\right)^{1000}$.

Now the $(n+1)$th term = the nth term $\times \frac{1000-n+1}{n}\frac{1}{55}$

And therefore the $(n+1)$th term > the nth term

$$\begin{aligned} \text{so long as} \quad 1000-n+1 &> 55n \\ \text{or} \quad 56n &< 1001 \\ \text{or} \quad n &< 17\tfrac{49}{56}. \end{aligned}$$

That is, the 18th term is the greatest.

But the 18th term involves 17 deaths; therefore the most probable number of deaths is 17.

2. Prove that upon De Moivre's hypothesis the force of mortality is equal to the rate of mortality at all ages.

$$\begin{aligned} \mu_x &= -\frac{1}{l_x}\frac{dl_x}{dx} \\ &= -\frac{1}{86-x}\frac{d(86-x)}{dx} \quad \text{(by the hypothesis)} \\ &= \frac{1}{86-x} \\ &= q_x \end{aligned}$$

3. If the force of mortality varies inversely as the complement of life, deduce the forms of l_x and $\mathring{e}_x$.

$$\mu_x \propto \frac{1}{\omega - x}$$

$$\text{Let} \quad \mu_x = \frac{k}{\omega - x}$$

$$\text{But} \quad \mu_x = -\frac{d \log l_x}{dx}$$

$$\text{Therefore} \quad \log l_x = -\int \mu_x dx$$

$$= -\int \frac{k}{\omega - x} dx$$

$$= k \log(\omega - x)$$

$$\text{whence} \quad l_x = (\omega - x)^k$$

$$\text{Also} \quad \mathring{e}_x = \int_0^{\omega - x} \frac{l_{x+t}}{l_x} dt$$

$$= \int_0^{\omega - x} \frac{(\omega - x - t)^k}{(\omega - x)^k} dt$$

$$\text{But} \quad \int \frac{(\omega - x - t)^k}{(\omega - x)^k} dt = \frac{1}{(\omega - x)^k} \frac{-(\omega - x - t)^{k+1}}{k+1}$$

$$\text{Therefore} \quad \mathring{e}_x = \frac{1}{(\omega - x)^k} \frac{(\omega - x)^{k+1}}{k+1}$$

$$= \frac{\omega - x}{k+1}$$

4. Obtain an expression for the force of mortality when the law of mortality is expressed by the equation

$$l_x = ks^x w^{x^2} g^{c^x}$$

$$\text{Since} \quad l_x = ks^x w^{x^2} g^{c^x}$$

$$\log l_x = \log k + x \log s + x^2 \log w + c^x \log g$$

$$\text{But} \quad \mu_x = -\frac{d \log l_x}{dx}$$

$$= -(\log s + 2x \log w + c^x \log c \log g)$$

$$\text{Now let} \quad -\log s = \text{A}$$

$$-2 \log w = \text{H}$$

$$\text{and} \quad -\log c \log g = \text{B}$$

$$\text{Then} \quad \mu_x = \text{A} + \text{H}x + \text{B}c^x$$

CHAPTER VII

Annuities and Assurances

1. By *Text Book* formula (3)

$$a_x = vp_x(1+a_{x+1})$$

whence

$$\begin{aligned}(1+i)a_x &= p_x(1+a_{x+1}) \\ &= (1+a_{x+1}) - q_x(1+a_{x+1})\end{aligned}$$

From the equation stated in this manner much that is useful may be learned.

If a_x be invested at age x in the purchase of a life annuity, by the end of one year the purchase-money will have accumulated to $(1+i)a_x$. Now if the annuitant has survived, he will be aged $(x+1)$, and will be entitled to the first payment of the annuity immediately. The value of the annuity will therefore be $(1+a_{x+1})$. The excess of $(1+a_{x+1})$ over $(1+i)a_x$ is, by the equation above, $q_x(1+a_{x+1})$, which is the loss to the granter of the annuity for each survivor, and therefore the total loss in respect of the survivors of l_x entrants at age x is $l_{x+1}q_x(1+a_{x+1})$. Again, if the annuitant dies during the first year the whole of the accumulation, that is $(1+i)a_x$, will be set free. The total amount set free by the deaths among l_x entrants at age x will be $d_x(1+i)a_x$. It is assumed that the mortality actually experienced will be that shown by the table adopted, and it may easily be proved that

$$l_{x+1}q_x(1+a_{x+1}) = d_x(1+i)a_x$$

Expressing *Text Book* formula (3) in slightly altered form, we have

$$l_x(1+i)a_x = l_{x+1}(1+a_{x+1})$$

From this it may be seen

(1) That, unless d_x die during the year, the accumulations of

the purchase-monies of l_x entrants will not be sufficient to meet the annuities payable to the survivors. This confirms Mr King's remarks in Article 4 that "we must always presuppose a sufficient number of such benefits to form an average; so that the contributions for those that never mature may be available to meet the deficiency in the contributions for those that actually become payable."

(2) That, when an annuitant dies, the reserve released is not to be considered profit to the office, since the amount released has to be applied to make good the deficiency under the contracts of those who survive.

2. The following simple proof shows the identity in result of Barrett's and Davies's forms of commutation columns. According to Barrett, who used the initial form,

$$
\begin{aligned}
a_x &= \frac{B_{x+1}}{A_x} \\
&= \frac{A_{x+1} + A_{x+2} + \cdots}{A_x} \\
&= \frac{(1+i)^{\omega-x-1} l_{x+1} + (1+i)^{\omega-x-2} l_{x+2} + \cdots}{(1+i)^{\omega-x} l_x} \\
&= \frac{(1+i)^{\omega} v^{x+1} l_{x+1} + (1+i)^{\omega} v^{x+2} l_{x+2} + \cdots}{(1+i)^{\omega} v^x l_x} \\
&= \frac{v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + \cdots}{v^x l_x} \\
&= \frac{D_{x+1} + D_{x+2} + \cdots}{D_x} \\
&= \frac{N_x}{D_x}
\end{aligned}
$$

according to Davies, who used the terminal form.

3. In *Text Book*, Article 25, it is shown that the value of a life annuity is less than the value of an annuity-certain for the term of the curtate expectation, and it may similarly be shown that the value of an assurance of 1 payable at the moment of death

is greater than the value of 1 due at the end of the term of the complete expectation.

$$\begin{aligned}
A_x &= 1 - d(1 + a_x) \\
&> 1 - d(1 + a_{\overline{e_x}|}), \text{ since } a_x < a_{\overline{e_x}|} \\
&> 1 - \frac{i}{1+i}(1+i)\frac{1 - v^{1+e_x}}{i} \\
&> v^{1+e_x} \\
&> v^{\mathring{e}_x + \kappa} \quad \text{where } 1 - \kappa \text{ is the time lived by } (x)
\end{aligned}$$

in the year of death.

Therefore $A_x(1+i)^\kappa > v^{\mathring{e}_x}$

or $\bar{A}_x > v^{\mathring{e}_x}$ since $\bar{A}_x = A_x(1+i)^\kappa$, κ being the time before the end of the year, at which death occurs.

Or we may prove it thus:—Let there be d_x quantities each equal to v, d_{x+1} each v^2, d_{x+2} each v^3, and so on. The arithmetic mean of these $d_x + d_{x+1} + d_{x+2} + \cdots$ quantities is

$$\frac{vd_x + v^2 d_{x+1} + v^3 d_{x+2} + \cdots}{d_x + d_{x+1} + d_{x+2} + \cdots}$$
$$= A_x$$

and their geometric mean is

$$v^{\frac{d_x + 2d_{x+1} + 3d_{x+2} + \cdots}{d_x + d_{x+1} + d_{x+2} + \cdots}}$$
$$= v^{1+e_x}$$

But the arithmetic mean of any number of positive quantities which are not all equal is always greater than their geometric mean, as proved on page 6.

Therefore $A_x > v^{1+e_x}$

and $\bar{A}_x > v^{\mathring{e}_x}$ as before.

$$\text{Again} \quad {}^{(\infty)}P_x = \frac{\bar{A}_x}{1 + a_x}$$
$$> \frac{v^{\mathring{e}_x}}{1 + a_{\overline{e_x}|}}$$

since $\bar{A}_x > v^{\mathring{e}_x}$ and $a_x < a_{\overline{e_x}|}$

From this we argue that the annual premium required to provide a payment of 1 at the moment of death is greater than the annual

premium payable in advance required to provide 1 at the end of the term-certain of the complete expectation of life.

4. If we are given any two of the three functions a_x, A_x, and P_x we can find i, the rate at which they are calculated.

(1) Given a_x and A_x.

$$\text{By } \textit{Text Book} \text{ formula (22) } A_x = \frac{1 - ia_x}{1+i}$$

$$\text{whence} \quad (1+i)A_x = 1 - ia_x$$

$$\text{and} \quad i = \frac{1 - A_x}{a_x + A_x}$$

(2) Given A_x and P_x.

$$\text{By } \textit{Text Book} \text{ formula (37) } P_x = \frac{dA_x}{1 - A_x}$$

$$\text{whence} \quad (1+i)P_x(1 - A_x) = iA_x$$

$$\text{and} \quad i = \frac{P_x(1 - A_x)}{A_x - P_x(1 - A_x)}$$

(3) Given P_x and a_x.

$$\text{By } \textit{Text Book} \text{ formula (38) } P_x = \frac{1}{1+a_x} - d$$

$$\text{whence} \quad (1+i)P_x(1+a_x) = 1 - ia_x$$

$$\text{and} \quad i = \frac{1 - P_x(1 + a_x)}{a_x + P_x(1 + a_x)}$$

5. The form of the equations given in *Text Book*, Article 39, can be adapted to the case of endowment assurances in every instance.

$$\begin{aligned} A_{\overline{xn|}} &= v(1 + a_{x:\overline{n-1|}}) - a_{x:\overline{n-1|}} \\ &= \frac{1 - ia_{x:\overline{n-1|}}}{1+i} \\ &= v - iva_{x:\overline{n-1|}} \\ &= 1 - d(1 + a_{x:\overline{n-1|}}) \\ &= \frac{a_\infty - a_{x:\overline{n-1|}}}{1 + a_\infty} \end{aligned}$$

All through $a_{x:\overline{n-1|}}$ is substituted for a_x, and it will be found a

useful exercise to reason these equations out for endowment assurances as is done in *Text Book*, Articles 30, and 41 to 44, for whole-life assurances.

6. In connection with *Text Book* formula (41), it should be noted that

$$\begin{aligned} a_{\overline{xn}|} &= \frac{D_{x+1}+D_{x+2}+\cdot\cdot\cdot+D_{x+n}}{D_x} \\ &= \frac{D_{x+1}+D_{x+2}+\cdot\cdot\cdot+D_{x+n-1}}{D_x} + \frac{D_{x+n}}{D_x} \\ &= a_{x:\overline{n-1}|} + A_{\overline{xn}|}^{\;1} \end{aligned}$$

Hence $\qquad A_{\overline{xn}|}^{\;1} = a_{\overline{xn}|} - a_{x:\overline{n-1}|}$

and $\qquad \begin{aligned} P_{\overline{xn}|}^{\;1} &= \frac{A_{\overline{xn}|}^{\;1}}{1+a_{x:\overline{n-1}|}} \\ &= \frac{a_{\overline{xn}|} - a_{x:\overline{n-1}|}}{1+a_{x:\overline{n-1}|}} \end{aligned}$

This formula is frequently useful and exhibits the method of finding the annual premium for a pure endowment by means of tables of temporary annuities alone.

7. By application of *Text Book* formula (57) we have

$$a_{\overline{xy}} = a_x + a_y - a_{xy}$$

And also by *Text Book* formula (63)

$$\begin{aligned} A_{\overline{xy}} &= v(1+a_{\overline{xy}}) - a_{\overline{xy}} \\ &= \{v(1+a_x)-a_x\} + \{v(1+a_y)-a_y\} - \{v(1+a_{xy})-a_{xy}\} \\ &= A_x + A_y - A_{xy} \end{aligned}$$

But it must be observed that we *cannot write*

$$P_{\overline{xy}} = P_x + P_y - P_{xy}$$

which will be at once apparent when it is remembered that

$$\begin{aligned} P_{\overline{xy}} &= \frac{A_{\overline{xy}}}{1+a_{\overline{xy}}} \\ &= \frac{A_x+A_y-A_{xy}}{1+a_{\overline{xy}}} \\ &= \frac{A_x}{1+a_{\overline{xy}}} + \frac{A_y}{1+a_{\overline{xy}}} - \frac{A_{xy}}{1+a_{\overline{xy}}} \end{aligned}$$

8. In *Text Book*, Article 88, $A_{\overline{wxyz\ldots(m)}}^{\;r}$ is described as an assurance on the last r survivors of m lives, by which is meant that it is an assurance payable at the end of the year in which at least r lives cease to survive, *i.e.*, the year in which occurs the death of the $(m-r+1)$th person. Now the probability that the $(m-r+1)$th death will occur in the nth year is

$$_{n-1|}q_{\overline{wxyz\ldots(m)}}^{\;r} = \left({}_{n-1}p_{\overline{wxyz\ldots(m)}}^{\;r} - {}_{n}p_{\overline{wxyz\ldots(m)}}^{\;r}\right).$$

Therefore
$$A_{\overline{wxyz\ldots(m)}}^{\;r} = \Sigma v^n\left({}_{n-1}p_{\overline{wxyz\ldots(m)}}^{\;r} - {}_{n}p_{\overline{wxyz\ldots(m)}}^{\;r}\right)$$
$$= v\left(1+\Sigma v^n\,{}_{n}p_{\overline{wxyz\ldots(m)}}^{\;r}\right) - \Sigma v^n\,{}_{n}p_{\overline{wxyz\ldots(m)}}^{\;r}$$
$$= v\left(1+a_{\overline{wxyz\ldots(m)}}^{\;r}\right) - a_{\overline{wxyz\ldots(m)}}^{\;r}$$

But again
$$A_{\overline{wxyz\ldots(m)}}^{\;r} = \Sigma v^n\,{}_{n-1|}q_{\overline{wxyz\ldots(m)}}^{\;r}$$
$$= \Sigma v^n\left\{Z^r - rZ^{r+1} + \frac{r(r+1)}{2}\,Z^{r+2} - \ldots\right\}$$

where each power of Z represents the sum of certain probabilities of the form $_{n-1|}q_{wxyz\ldots(t)}$, the number t being the same as the index of the particular power of Z involved. We may therefore extend the meaning of Z for use with assurances; so that Z^r may signify the sum of the values of the assurances on r joint lives for all the combinations of r lives that can be made out of m lives. We shall then have

$$A_{\overline{wxyz\ldots(m)}}^{\;r} = Z^r - rZ^{r+1} + \frac{r(r+1)}{2}\,Z^{r+2} - \ldots$$

Taking as examples:—

$$A_{\overline{wxyz}}^{\;1} = Z - Z^2 + Z^3 - Z^4$$
$$= (A_w + A_x + A_y + A_z) - (A_{wx} + A_{wy} + A_{wz} + A_{xy} + A_{xz} + A_{yz})$$
$$+ (A_{wxy} + A_{wxz} + A_{wyz} + A_{xyz}) - A_{wxyz}$$

$$A_{\overline{wxyz}}^{\;2} = Z^2 - 2Z^3 + 3Z^4$$
$$= (A_{wx} + A_{wy} + A_{wz} + A_{xy} + A_{xz} + A_{yz})$$
$$- 2(A_{wxy} + A_{wxz} + A_{wyz} + A_{xyz}) + 3A_{wxyz}$$

$$A_{\overline{wxyz}}^{\;3} = Z^3 - 3Z^4$$
$$= (A_{wxy} + A_{wxz} + A_{wyz} + A_{xyz}) - 3A_{wxyz}$$

$$A_{\overline{wxyz}}^{\;4} = Z^4 = A_{wxyz}$$

9. The *Text Book* skeleton formula (57) can be applied to temporary benefits and deferred benefits, by altering the meaning of Z.

Thus we may say

$$_{\overline{n|}}a_{\overline{wxyz\ldots(m)}}^{\;r} = Z^r - rZ^{r+1} + \frac{r(r+1)}{2}Z^{r+2} - \ldots$$

where Z^r signifies the sum of the values of the temporary annuities on r joint lives for n years, for all the combinations of r lives that can be made out of m lives.

Or again $${}_{n|}A_{\overline{wxyz\ldots(m)}}^{\;r} = Z^r - rZ^{r+1} + \frac{r(r+1)}{2}Z^{r+2} - \ldots$$

where Z^r signifies the sum of the values of the assurances deferred n years on r joint lives for all the combinations of r lives that can be made out of m lives.

10. To find the present value at rate i of the amount at rate j of an annuity-due of 1 per annum to accumulate during the lifetime of (x).

The following is perhaps a clearer demonstration than that given in *Text Book*, Article 98.

If (x) die in the first year the amount of accumulated annuity payable is $(1+j) = (1+j)s_{\overline{1|}}$.

If (x) die in the second year the amount payable is

$$\{(1+j)^2 + (1+j)\} = (1+j)s_{\overline{2|}}.$$

And so on, and generally if (x) die in the tth year the amount payable is $(1+j)s_{\overline{t|}}$.

The present value is to be taken at rate i, and the probability of (x)'s death in the tth year is $\frac{d_{x+t-1}}{l_x}$. Therefore the present value of the amount to be paid in the event of (x)'s death in the tth year is

$$v^t(1+j)\frac{(1+j)^t - 1}{j}\;\frac{d_{x+t-1}}{l_x}$$

$$\text{Hence} \quad A = \Sigma_{t=1}^{t=\omega-x} v^t(1+j)\frac{(1+j)^t-1}{j}\frac{d_{x+t-1}}{l_x}$$

$$= \frac{1+j}{j}\left\{\Sigma v^t(1+j)^t\frac{d_{x+t-1}}{l_x} - \Sigma v^t\frac{d_{x+t-1}}{l_x}\right\}$$

$$= \frac{1+j}{j}(A'_x - A_x)$$

where A'_x is calculated at rate J, which is such that $\frac{1+j}{1+i} = \frac{1}{1+J}$.

For an alternative solution we have

$$A = v(1+j)\frac{d_x}{l_x} + v^2\left\{(1+j)+(1+j)^2\right\}\frac{d_{x+1}}{l_x}$$

$$+ v^3\left\{(1+j)+(1+j)^2+(1+j)^3\right\}\frac{d_{x+2}}{l_x} + \dots$$

$$= (1+j)\left(\frac{vd_x + v^2d_{x+1} + v^3d_{x+2} + \dots}{l_x}\right) + (1+j)^2\left(\frac{v^2d_{x+1} + v^3d_{x+2} + \dots}{l_x}\right)$$

$$+ (1+j)^3\left(\frac{v^3d_{x+2} + \dots}{l_x}\right) + \dots$$

$$= \frac{(1+j)M_x + (1+j)^2M_{x+1} + (1+j)^3M_{x+2} + \dots}{D_x}$$

11. To find the present value at rate i of the amount at rate j of a temporary annuity-due of 1 per annum for n years to accumulate during the lifetime of (x), the accumulations to be payable only in the event of (x)'s death within that period.

As in the previous problem, the present value of the amount to be paid in the event of (x)'s death in the tth year is

$$v^t(1+j)\frac{(1+j)^t-1}{j}\frac{d_{x+t-1}}{l_x}$$

$$\text{Hence} \quad A = \Sigma_{t=1}^{t=n} v^t(1+j)\frac{(1+j)^t-1}{j}\frac{d_{x+t-1}}{l_x}$$

$$= \frac{1+j}{j}\left\{\Sigma_1^n v^t(1+j)^t\frac{d_{x+t-1}}{l_x} - \Sigma_1^n v^t\frac{d_{x+t-1}}{l_x}\right\}$$

$$= \frac{1+j}{j}(A'^{1}_{x\overline{n}|} - A^{1}_{x\overline{n}|})$$

where $A'^{1}_{x\overline{n}|}$ is calculated at rate J, which is such that $\frac{1+j}{1+i} = \frac{1}{1+J}$.

According to the alternative method

$$A = v(1+j)\frac{d_x}{l_x} + v^2\left\{(1+j)+(1+j)^2\right\}\frac{d_{x+1}}{l_x} + \ldots$$

$$+ v^n\left\{(1+j)+(1+j)^2+\cdots+(1+j)^n\right\}\frac{d_{x+n-1}}{l_x}$$

$$= (1+j)\left(\frac{vd_x + v^2d_{x+1} + \cdots + v^n d_{x+n-1}}{l_x}\right) + (1+j)^2\left(\frac{v^2d_{x+1} + \cdots + v^n d_{x+n-1}}{l_x}\right)$$

$$+ \cdot \cdot \cdot + (1+j)^n \frac{v^n d_{x+n-1}}{l_x}$$

$$= \frac{(1+j)(M_x - M_{x+n}) + (1+j)^2(M_{x+1} - M_{x+n}) + \cdots + (1+j)^n(M_{x+n-1} - M_{x+n})}{D_x}$$

12. A different problem from the last is to find the present value at rate i of the amount at rate j of an annuity-due of 1 per annum which is to be allowed to accumulate until all of l_x persons are dead.

This means that each payment is to be allowed to accumulate at rate j up to the limit of life, and the whole must be discounted from that time at rate i. Therefore

$$A = v^{\omega-x} \times \frac{l_x(1+j)^{\omega-x} + l_{x+1}(1+j)^{\omega-x-1} + \cdots + l_{\omega-1}(1+j)}{l_x}$$

$$= \left(\frac{1+j}{1+i}\right)^{\omega-x} \times \frac{l_x + l_{x+1}(1+j)^{-1} + \cdots + l_{\omega-1}(1+j)^{-(\omega-x-1)}}{l_x}$$

$$= \left(\frac{1+j}{1+i}\right)^{\omega-x}(1+a'_x)$$

where a'_x is calculated at rate j.

For the value of a temporary annuity-due which is to be allowed to accumulate to the end of the term under similar conditions, we have

$$A = v^n \frac{l_x(1+j)^n + l_{x+1}(1+j)^{n-1} + \cdot \cdot \cdot + l_{x+n-1}(1+j)}{l_x}$$

$$= \left(\frac{1+j}{1+i}\right)^n \times \frac{l_x + l_{x+1}(1+j)^{-1} + \cdot \cdot \cdot + l_{x+n-1}(1+j)^{-(n-1)}}{l_x}$$

$$= \left(\frac{1+j}{1+i}\right)^n (1 + a'_{x:\overline{n-1}|}) \text{ where } a'_{x:\overline{n-1}|} \text{ is calculated at rate } j.$$

13. To find the value of a temporary assurance for n years on the life of (x), commencing at 1 and increasing by 1 per annum so long as (x) and (y) are jointly alive.

According to the terms of the contract it will be noticed that 1 is payable in any case provided (x) dies between ages x and $x+n$. The value of this portion is therefore $\dfrac{M_x - M_{x+n}}{D_x}$. This sum assured of 1 falls to be increased by 1 provided (y) lives one year and (x) dies between ages $x+1$ and $x+n$. The value of this second portion is therefore $p_y\dfrac{M_{x+1} - M_{x+n}}{D_x}$. And so on for every year until the nth, the last increase taking place if (y) lives $n-1$ years and (x) dies between ages $x+n-1$ and $x+n$, the value thereof being ${}_{n-1}p_y\dfrac{M_{x+n-1} - M_{x+n}}{D_x}$.

Therefore the whole value of the assurance is

$$A = \frac{1}{D_x}\Big\{(M_x - M_{x+n}) + p_y(M_{x+1} - M_{x+n}) + \cdots + {}_{n-1}p_y(M_{x+n-1} - M_{x+n})\Big\}$$

14. The annuity of *Text Book*, Article 99, is payable so long as (x) lives, but not more than t years after the death of (y). Now as to the first t years, it is obvious that (y) does not come into consideration at all; but, in order that a payment may be made on (x)'s surviving the $(t+1)$th year, it is necessary that (y) should have lived at least one year, and similarly for following years. The value of the whole annuity is therefore

$$\frac{vl_{x+1} + v^2l_{x+2} + \cdots + v^tl_{x+t}}{l_x} + \frac{v^{t+1}\,l_{x+t+1}\,l_{y+1} + v^{t+2}\,l_{x+t+2}\,l_{y+2} + \cdots}{l_x\,l_y}$$

$$= a_{x\overline{t}|} + \frac{v^t\,l_{x+t}}{l_x}\cdot\frac{vl_{x+t+1}\,l_{y+1} + v^2l_{x+t+2}\,l_{y+2} + \cdots}{l_{x+t}\,l_y}$$

$$= a_{x\overline{t}|} + v^t\,{}_tp_x\,a_{x+t:y}$$

$$= a_x - \frac{D_{x+t}}{D_x}(a_{x+t} - a_{x+t:y})$$

15. To find the value of an annuity to (x), the first payment

to be made at the end of the tth year succeeding the year in which (y) dies.

The value of this annuity could be obtained by deducting from a_x the foregoing annuity, for this would give the value of a life annuity to (x) less that of an annuity payable so long as (x) lives jointly with (y) and for t years after the death of (y), should (x) live so long. The difference is obviously the value of the desired annuity.

Thus
$$\begin{aligned}
a &= a_x - a_{x:\overline{y(\overline{t}|)}} \\
&= a_x - \left\{ a_x - \frac{D_{x+t}}{D_x}(a_{x+t} - a_{x+t:y}) \right\} \\
&= \frac{D_{x+t}}{D_x}(a_{x+t} - a_{x+t:y})
\end{aligned}$$

Or we may proceed as follows :—

$$\begin{aligned}
a &= \Sigma_{n=1}^{n=\omega-x} v^n \frac{d_{y+n-1}}{l_y} \frac{l_{x+n}}{l_x} \times {}_{t|}a_{x+n} \\
&= \Sigma v^{n+t} \frac{l_{y+n-1} - l_{y+n}}{l_y} \frac{l_{x+n+t}}{l_x}(1 + a_{x+n+t}) \\
&= \frac{D_{x+t}}{D_x} \Sigma \frac{l_{y+n-1} - l_{y+n}}{l_y} \frac{v^n l_{x+n+t} + v^{n+1} l_{x+n+t+1} + \cdot \cdot \cdot}{l_{x+t}} \\
&= \frac{D_{x+t}}{D_x} \left\{ \frac{l_y - l_{y+1}}{l_y}(vp_{x+t} + v^2{}_2p_{x+t} + v^3{}_3p_{x+t} + \cdot \cdot \cdot) \right. \\
&\qquad + \frac{l_{y+1} - l_{y+2}}{l_y}(v^2{}_2p_{x+t} + v^3{}_3p_{x+t} + \cdot \cdot \cdot) \\
&\qquad + \frac{l_{y+2} - l_{y+3}}{l_y}(v^3{}_3p_{x+t} + \cdot \cdot \cdot) \\
&\qquad \left. + \cdot \cdot \cdot \right\} \\
&= \frac{D_{x+t}}{D_x} \left\{ (vp_{x+t} + v^2{}_2p_{x+t} + v^3{}_3p_{x+t} + \cdot \cdot \cdot) \right. \\
&\qquad \left. - (vp_{x+t} \times p_y + v^2{}_2p_{x+t} \times {}_2p_y + v^3{}_3p_{x+t} \times {}_3p_y + \cdot \cdot \cdot) \right\} \\
&= \frac{D_{x+t}}{D_x}(a_{x+t} - a_{x+t:y})
\end{aligned}$$

16. To find the value of an annuity payable for t years certain after the death of (y), and thereafter so long as a life presently aged (x) may live.

The value of the first t payments of this annuity is that of an annuity-certain of which the first payment is made at the end of the year of death of (y), or $A_y(1+a_{\overline{t-1}|})$. The value of the subsequent payments is that of an annuity to (x), of which the first payment is made at the end of the tth year succeeding the year of death of (y), and of which the value as found above is $\frac{D_{x+t}}{D_x}(a_{x+t}-a_{x+t:y})$.

The whole value is therefore

$$A_y(1+a_{\overline{t-1}|}) + \frac{D_{x+t}}{D_x}(a_{x+t}-a_{x+t:y})$$

17. To find the value of an annuity payable so long as (x) lives with (y) and for n years after the death of (y), but in any event no payment to be made after m years from the present time, m being greater than n.

The value of this annuity is

$$\frac{vl_{x+1}+v^2l_{x+2}+\cdot\cdot\cdot+v^n l_{x+n}}{l_x}$$
$$+\frac{v^{n+1}l_{x+n+1}\,l_{y+1}+v^{n+2}l_{x+n+2}\,l_{y+2}+\cdots+v^m l_{x+m}\,l_{y+m-n}}{l_x l_y}$$
$$=a_{x\overline{n}|}+\frac{D_{x+n}}{D_x}a_{x+n:y:\overline{m-n}|}$$

18. To find the value of an annuity payable so long as the survivor of (y) and (z) lives with (x) and for t years after the death of (x), should the survivor of (y) and (z) live so long.

Following *Text Book*, Article 99, we have

$$\begin{aligned}
a_{\overline{yz}:\overline{x(\overline{t}|)}} &= a_{y:\overline{x(\overline{t}|)}}+a_{z:\overline{x(\overline{t}|)}}-a_{yz:\overline{x(\overline{t}|)}}\\
&= a_y-\frac{D_{y+t}}{D_y}(a_{y+t}-a_{x:y+t})\\
&\quad +a_z-\frac{D_{z+t}}{D_z}(a_{z+t}-a_{x:z+t})\\
&\quad -a_{yz}+\frac{D_{y+t:z+t}}{D_{yz}}(a_{y+t:z+t}-a_{x:y+t:z+t})
\end{aligned}$$

19. To find the value of an annuity payable so long as (x) lives with the survivor of (y) and (z), and for t years after the survivor's death should (x) live so long.

$$\begin{aligned}
a_{x:\overline{\overline{yz}(\overline{t}|)}} &= a_{x:\overline{y(\overline{t}|)}} + a_{x:\overline{z(\overline{t}|)}} - a_{x:\overline{yz(\overline{t}|)}} \\
&= a_x - \frac{D_{x+t}}{D_x}(a_{x+t} - a_{x+t:y}) \\
&\quad + a_x - \frac{D_{x+t}}{D_x}(a_{x+t} - a_{x+t:z}) \\
&\quad - a_x + \frac{D_{x+t}}{D_x}(a_{x+t} - a_{x+t:yz}) \\
&= a_x - \frac{D_{x+t}}{D_x}(a_{x+t} - a_{x+t:y} - a_{x+t:z} + a_{x+t:yz}) \\
&= a_x - \frac{D_{x+t}}{D_x}(a_{x+t} - a_{x+t:\overline{yz}})
\end{aligned}$$

20. If we are given a table of P_x at a certain rate of interest we may deduce the mortality table. The process is as set forth in the following schedule:—

Age x	P_x	a_x	λa_x	$\lambda(1+a_{x+1})$	(4)-(5)	(6) $+\lambda(1+i)$ $=\lambda p_x$	λl_x	l_x
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)

The values in (3) are obtained from those in (2) by entering conversion tables inversely as explained fully in Chapter VIII. The value in (7) is

$$\begin{aligned}
\log a_x - \log(1+a_{x+1}) + \log(1+i) &= \log\frac{(1+i)a_x}{1+a_{x+1}} \\
&= \log p_x
\end{aligned}$$

As usual we fix a radix or initial value of l_x the log of which is placed at the head of (8), whence by continued addition of the successive values in (7) those in (8) are found. The values in (9) are the natural numbers corresponding to the logs in (8). Here we may repeat what is said in *Text Book*, Chapter III., Article 18, viz., "The values in the column of l_x so formed will not be the same as in the corresponding column from which the given table (of P_x) was originally calculated, unless the radix we choose is the same as in that table; but the *ratios* between the values will be the same, and that is all that is required. As before remarked, the column l_x does not give absolute numbers living, but only relative numbers."

It may be mentioned that an easy way of obtaining *Text Book* formula (72) is from formula (70). Thus by formula (70) we have

$$p_x = \frac{(1+i)a_x}{1+a_{x+1}}$$

But by *Text Book* formula (39)

$$a_x = \frac{1}{P_x+d} - 1$$

and therefore

$$\frac{1}{1+a_{x+1}} = P_{x+1}+d$$

Substituting these two results in formula (70) we have

$$p_x = (1+i)(P_{x+1}+d)\left(\frac{1}{P_x+d} - 1\right)$$

21. By the use of the theory of varying annuities (*Theory of Finance*, Chapter III.) the values of the functions a_x, A_{xy}, and a_{xy} on De Moivre's hypothesis may very easily be found.

$$a_x = \frac{v(n-1)+v^2(n-2)+\cdot\cdot\cdot+v^n(n-n)}{n}$$

$$= \frac{(n-1)a_{\overline{n}|\overline{1}|} - a_{\overline{n}|\overline{2}|}}{n}$$

$$= \frac{(n-1)a_{\overline{n}|} - \dfrac{a_{\overline{n}|} - nv^n}{i}}{n} \quad \text{since } a_{\overline{n}|\overline{r}|} = \frac{a_{\overline{n}|\,\overline{r-1}|} - v^n l_{\overline{n+1}|\,\overline{r}|}}{i}$$

$$a_x = \frac{(n-1)(1-v^n)-(a_{\overline{n}|}-nv^n)}{ni}$$

$$= \frac{n-(1+a_{\overline{n-1}|})}{ni}$$

$$= \frac{n-(1+i)a_{\overline{n}|}}{ni}$$

Again, from the reasoning in *Text Book*, Article 108, we may form a mortality table for joint lives as follows:—

Ages.	Couples remaining.	Couples broken.
xy	nm	$n+m-1$
$x+1:y+1$	$(n-1)(m-1)$	$n+m-3$
$x+2:y+2$	$(n-2)(m-2)$	$n+m-5$
$x+3:y+3$	$(n-3)(m-3)$	$n+m-7$
etc.	etc.	etc.

Hence

$$A_{xy} = \frac{v(n+m-1)+v^2(n+m-3)+v^3(n+m-5)+\cdots \text{ to } n \text{ terms}}{nm}$$

$$= \frac{1}{nm}\left\{(n+m-1)a_{\overline{n}|\overline{1}|}-2a_{\overline{n}|\overline{2}|}\right\}$$

$$= \frac{1}{nm}\left\{(n+m-1)a_{\overline{n}|}-2\,\frac{a_{\overline{n}|}-nv^n}{i}\right\}$$

$$\text{Also } a_{xy} = \frac{1}{nm}\Big[v\{nm-(n+m)+1\}+v^2\{nm-2(n+m)+4\}$$

$$+v^3\{nm-3(n+m)+9\}+\cdots \text{ to } n \text{ terms}\Big]$$

$$= \frac{1}{nm}\Big[\{nm-(n+m)+1\}a_{\overline{n}|\overline{1}|}-(n+m-3)a_{\overline{n}|\overline{2}|}+2a_{\overline{n}|\overline{3}|}\Big]$$

and $a_{\overline{n}|\overline{1}|} = a_{\overline{n}|}$

$$a_{\overline{n}|\overline{2}|} = \frac{a_{\overline{n}|}-nv^n}{i}$$

$$a_{\overline{n}|\overline{3}|} = \frac{a_{\overline{n}|\overline{2}|}-\frac{n(n-1)}{2}v^n}{i}$$

22. In *Text Book*, Article 115, we have the value of a temporary annuity on two lives, the term varying with each life. Similarly we may have a deferred annuity on two lives, the period for which it is deferred varying with each life. For example, the annuity payable till the survivor of two lives, aged six and eleven respectively, attains majority is

$$|_{15}a_6 + |_{10}a_{11} - |_{10}a_{6:11}$$

And by analogy the annuity payable to the survivor of two lives, aged six and eleven respectively, but deferred till each or the survivor has attained majority, is

$$_{15}|a_6 + {}_{10}|a_{11} - {}_{15}|a_{6:11}$$

The general formula for such an annuity is

$$_n|a_a + {}_m|a_x - {}_n|a_{ax} \quad \text{if } m < n$$

or $$_n|a_a + {}_m|a_x - {}_m|a_{ax} \quad \text{if } m > n.$$

23. Again, the assurance payable if either of two lives, aged six and eleven respectively, dies before attaining majority, is

$$A^{1}_{(6:\overline{15}|)}{}^{1}_{(11:\overline{10}|)} = A^{\;1}_{\overline{6:11}:\overline{10}|} + \frac{D_{16:21}}{D_{6:11}} A^{1}_{16:\overline{5}|}$$

and for the annual premium we shall have

$$P^{1}_{(6:\overline{15}|)}{}^{1}_{(11:\overline{10}|)} = \frac{A^{\;1}_{\overline{6:11}:\overline{10}|} + \dfrac{D_{16:21}}{D_{6:11}} A^{1}_{16:\overline{5}|}}{a_{6:11:\overline{10}|} + \dfrac{D_{16:21}}{D_{6:11}} a_{16\cdot\overline{5}|}}$$

Generally

$$P^{1}_{(a\overline{n}|)}{}^{1}_{(x\overline{m}|)} = \frac{A^{\;1}_{\overline{ax}\,\overline{m}|} + \dfrac{D_{a+m:x+m}}{D_{ax}} A^{\;1}_{a+m:\overline{n-m}|}}{a_{ax\overline{m}|} + \dfrac{D_{a+m:x+m}}{D_{ax}} a_{a+m:\overline{n-m}|}} \quad \text{if } m < n$$

$$\text{or} \quad = \frac{A^{\;1}_{\overline{ax}\,\overline{n}|} + \dfrac{D_{a+n:x+n}}{D_{ax}} A^{\;1}_{x+n:\overline{m-n}|}}{a_{ax\overline{n}|} + \dfrac{D_{a+n:x+n}}{D_{ax}} a_{x+n:\overline{m-n}|}} \quad \text{if } m > n$$

24. Also the assurance payable if both lives, (six) and (eleven), die before attaining majority is

$$A_{\overline{(\overset{1}{6:\overline{15|}})(\overset{1}{11:\overline{10|}})}} = A^1_{6:\overline{15|}} + A^1_{11:\overline{10|}} - A_{(\overset{1}{6:\overline{15|}})(\overset{1}{11:\overline{10|}})}$$

$$= A^1_{6:\overline{15|}} + A^1_{11:\overline{10|}} - A_{\overset{1}{\overline{6:11}}:\overline{10|}} - \frac{D_{16:21}}{D_{6:11}} A^1_{16:\overline{5|}}$$

The annual premium for this benefit is

$$P_{\overline{(\overset{1}{6:\overline{15|}})(\overset{1}{11:\overline{10|}})}} = \frac{A_{\overline{(\overset{1}{6:\overline{15|}})(\overset{1}{11:\overline{10|}})}}}{a_{\overline{(6:\overline{15|})(11:\overline{10|})}} - \frac{D_{16:21}}{D_{6:11}} a_{16:\overline{5|}}}$$

$$= \frac{A^1_{6:\overline{15|}} + A^1_{11:\overline{10|}} - A_{\overset{1}{\overline{6:11}}:\overline{10|}} - \frac{D_{16:21}}{D_{6:11}} A^1_{16:\overline{5|}}}{a_{6:\overline{15|}} + a_{11:\overline{10|}} - a_{6:11:\overline{10|}} - \frac{D_{16:21}}{D_{6:11}} a_{16:\overline{5|}}}$$

Generally

$$P_{\overline{(\overset{1}{a\overline{n|}})(\overset{1}{x\overline{m|}})}} = \frac{A^1_{a\overline{n|}} + A^1_{x\overline{m|}} - A_{\overset{1}{\overline{ax}}\,\overline{m|}} - \frac{D_{a+m:x+m}}{D_{ax}} A_{\overset{1}{a+m}:\overline{n-m|}}}{a_{a\overline{n|}} + a_{x\overline{m|}} - a_{ax\overline{m|}} - \frac{D_{a+m:x+m}}{D_{ax}} a_{a+m:\overline{n-m|}}} \quad \text{if } m < n$$

$$\text{or} \quad = \frac{A^1_{a\overline{n|}} + A^1_{x\overline{m|}} - A_{\overset{1}{\overline{ax}}\,\overline{n|}} - \frac{D_{a+n:x+n}}{D_{ax}} A_{\overset{1}{x+n}:\overline{m-n|}}}{a_{a\overline{n|}} + a_{x\overline{m|}} - a_{ax\overline{n|}} - \frac{D_{a+n:x+n}}{D_{ax}} a_{x+n:\overline{m-n|}}} \quad \text{if } m > n$$

25. The following are problems connected with national insurance. Given a stationary population where the numbers living at each age correspond with the figures in some known table of mortality, find

(*a*) The amount that will require to be subscribed to provide 1 at the death of each member of the community.

The deaths in the first year are $(d_0 + d_1 + d_2 + \dots + d_{\omega-1}) = l_0$

" " second " $(d_1 + d_2 + d_3 + \dots + d_{\omega-1}) = l_1$

" " third " $(d_2 + d_3 + d_4 + \dots + d_{\omega-1}) = l_2$

and so on.

The present value of the benefit is therefore

$$\begin{aligned}\Sigma_0^{\omega-1}\, l_x \mathrm{A}_x &= vl_0 + v^2 l_1 + v^3 l_2 + \cdot \cdot \cdot \\ &= vl_0(1+a_0) \\ &= l_0(a_0 + \mathrm{A}_0).\end{aligned}$$

If each member of the community is to contribute the same single premium irrespective of age, we must divide the value of the benefit by the population. The total population is

$$l_0 + l_1 + l_2 + \cdot \cdot \cdot = l_0(1+e_0).$$

Therefore the single premium that each must contribute is

$$\frac{vl_0(1+a_0)}{l_0(1+e_0)} = \frac{v(1+a_0)}{1+e_0}$$

(*b*) The fund that will require to be subscribed to pay an annuity-due of 1 per annum to every member of the community.

$$\begin{aligned}\Sigma_0^{\omega-1}\, l_x(1+a_x) &= l_0(1+a_0) + l_1(1+a_1) + l_2(1+a_2) + \cdot \cdot \cdot \\ &= l_0\frac{1-\mathrm{A}_0}{d} + l_1\frac{1-\mathrm{A}_1}{d} + l_2\frac{1-\mathrm{A}_2}{d} + \cdot \cdot \cdot \\ &= \frac{1}{d}\left\{(l_0 + l_1 + l_2 + \cdot \cdot \cdot) - \Sigma_0^{\omega-1}\, l_x \mathrm{A}_x\right\} \\ &= \frac{1}{d}\left\{l_0(1+e_0) - vl_0(1+a_0)\right\} \\ &= \frac{(1+i)l_0(1+e_0) - l_0(1+a_0)}{i}\end{aligned}$$

(*c*) The fund that will require to be subscribed to pay an annuity-due of 1 per annum to every member of the community; no payments to be made at and after age y.

The expression for this may be stated thus:—

$$\begin{aligned}&\Sigma_0^{\omega-1}\, l_x(1+a_x) - \Sigma_y^{\omega-1}\, l_x(1+a_x) \\ &\quad - \{l_{y-1}vp_{y-1}(1+a_y) + l_{y-2}v^2{}_2p_{y-2}(1+a_y) + \cdot \cdot \cdot \\ &\qquad + l_0 v^y{}_y p_0(1+a_y)\}\end{aligned}$$

that is, we deduct from the fund required in respect of the whole community that portion of it which is not required in respect of those who are at present aged y and upwards, and from the result

we deduct the value of the deferred annuities payable after attainment of age y to those who at present are of younger age. The expression may be reduced as follows :—

$$\Sigma_0^{y-1} l_x(1+a_x) - l_y(1+a_y)(v+v^2+v^3+ \cdot \cdot \cdot +v^y)$$

$$= \Sigma_0^{y-1} l_x(1+a_x) - l_y(1+a_y)\frac{1-v^y}{i}$$

$$= \Sigma_0^{y-1} l_x(1+a_x) - l_0\frac{(1+i)^y-1}{i}v^y{}_yp_0(1+a_y)$$

$$= \frac{(1+i)\{l_0(1+e_0)-l_y(1+e_y)\}-\{l_0(1+a_0)-l_y(1+a_y)\}}{i} - l_0 s_{\overline{y|}} \times {}_{y|}a_0$$

(*d*) The fund which will require to be subscribed to provide an old age pension of 1 per annum, first payment on attainment of age y.

The value of this fund is merely the last term in the preceding problem, that is

$$l_{y-1}vp_{y-1}(1+a_y)+l_{y-2}v^2{}_2p_{y-2}(1+a_y)+ \cdot \cdot \cdot$$
$$+l_0v^y{}_yp_0(1+a_y)$$
$$= l_0 s_{\overline{y|}} \times {}_{y|}a_0$$

(*e*) The annual premium, equal at all ages, to be subscribed by each member of the community to provide 1 at each death.

This premium will be obtained by dividing the fund of problem (*a*) by the expression found in problem (*b*), since the benefit side is $\Sigma_0^{\omega-1} l_xA_x$, and the payment side is $P \times \Sigma_0^{\omega-1} l_x(1+a_x)$.

(*f*) The annual premium equal at all ages, payable up to but not including age y, to provide an old age pension of 1 per annum, first payment on attaining age y.

This premium will be obtained by dividing the expression in problem (*d*) by that in problem (*c*).

These problems are discussed by Mr R. P. Hardy in a paper on "Collective Assurance" (*J.I.A.*, xxx. 79), where the formulas are given.

26. It is interesting to notice the forms taken by various benefits on the assumption that money yields no interest.

If $i=0$, $v=1$, and

$$a_x = \frac{vl_{x+1}+v^2l_{x+2}+v^3l_{x+3}+\cdots}{l_x}$$

$$= \frac{l_{x+1}+l_{x+2}+l_{x+3}+\cdots}{l_x}$$

$$= e_x$$

Also $$a_{\overline{xn}|} = \frac{l_{x+1}+l_{x+2}+l_{x+3}+\cdots+l_{x+n}}{l_x}$$

$$= e_{\overline{xn}|}$$

And $${}_{n|}a_x = {}_{n|}e_x$$

Again $$A_x = \frac{vd_x+v^2d_{x+1}+v^3d_{x+2}+\cdots}{l_x}$$

$$= \frac{d_x+d_{x+1}+d_{x+2}+\cdots}{l_x}$$

$$= 1$$

$$A^1_{\overline{xn}|} = \frac{d_x+d_{x+1}+\cdots+d_{x+n-1}}{l_x}$$

$$= \frac{l_x-l_{x+n}}{l_x}$$

$$= (1-{}_np_x)$$

$${}_{n|}A_x = A_x - A^1_{\overline{xn}|}$$

$$= 1-(1-{}_np_x)$$

$$= {}_np_x$$

$$A_{\overline{xn}|} = A^1_{\overline{xn}|} + A_{\overline{xn}|}^{\;\;1}$$

$$= (1-{}_np_x)+{}_np_x$$

$$= 1$$

$$P_x = \frac{A_x}{1+a_x}$$

$$= \frac{1}{1+e_x}$$

$$P^1_{\overline{xn}|} = \frac{A^1_{\overline{xn}|}}{1+a_{x:\overline{n-1}|}}$$

$$= \frac{1-{}_np_x}{1+e_{x:\overline{n-1}|}}$$

$$P({}_{n|}A_x) = \frac{{}_{n|}A_x}{1+a_x}$$

$$= \frac{{}_np_x}{1+e_x}$$

$$P_{\overline{xn}|} = \frac{A_{\overline{xn}|}}{1+a_{x:\overline{n-1}|}}$$

$$= \frac{1}{1+e_{x:\overline{n-1}|}}$$

Further $$(IA)_x = \frac{vd_x + 2v^2d_{x+1} + 3v^3d_{x+2} + \cdots}{l_x}$$

$$= \frac{d_x + 2d_{x+1} + 3d_{x+2} + \cdots}{l_x}$$

$$= \frac{l_x + l_{x+1} + l_{x+2} + \cdots}{l_x}$$

$$= 1 + e_x$$

27. Select tables of mortality supply us with the rates of mortality experienced in each year from entry amongst the entrants at each particular age. The notation is not difficult to grasp. For example,

$q_{[x]}$ is the rate of mortality experienced in the first year of insurance among those who enter at age x,

$q_{[x]+1}$ is the rate of mortality experienced in the second year of insurance among those who enter at age x, and generally

$q_{[x]+t}$ is the rate of mortality experienced in the $(t+1)$th year of insurance among those who enter at age x.

The suffix appended to the various symbols is formed by putting the age at which the life enters in square brackets, and adding the particular number of years after entry outside these brackets.

It is not usual to give for the several ages at entry the rates of mortality experienced during each year of insurance up to the limit of life; but only for a limited number of years (n), after

which the rates are joined to what has been called the "ultimate" experience. This ultimate table shows the rates of mortality amongst those who have been insured for n or more years, the assumption being that the mortality at age $x+n$ is the same for all of that age whether they entered at x or younger ages

This period of n years is not always assumed to be of the same length. For example, in Dr Sprague's Select Table ($H^{[M]}$) it is five years, while in the British Offices Life Tables, 1893 ($O^{[M]}$) it is taken at ten years.

The numbers living, in respect of age at entry x, after the several years of duration are expressed as follows:—

$$l_{[x]}, l_{[x]+1}, l_{[x]+2}, \dots l_{[x]+n-1}, l_{x+n}, l_{x+n+1}, \dots$$

The deaths similarly are denoted by

$$d_{[x]}, d_{[x]+1}, d_{[x]+2}, \dots d_{[x]+n-1}, d_{x+n}, d_{x+n+1}, \dots$$

For the annuity commutation columns we have

$$D_{[x]} = v^x l_{[x]}, \ D_{[x]+1} = v^{x+1} l_{[x]+1}, \dots D_{x+n} = v^{x+n} l_{x+n}, \dots$$
$$N_{[x]} = D_{[x]} + D_{[x]+1} + D_{[x]+2} + \dots + D_{[x]+n-1} + D_{x+n} + D_{x+n+1} + \dots$$

Also the assurance commutation columns are

$$C_{[x]} = v^{x+1} d_{[x]}, \ C_{[x]+1} = v^{x+2} d_{[x]+1}, \dots C_{x+n} = v^{x+n+1} d_{x+n}, \dots$$
$$M_{[x]} = C_{[x]} + C_{[x]+1} + C_{[x]+2} + \dots + C_{[x]+n-1} + C_{x+n} + C_{x+n+1} + \dots$$

28. It will be useful to discuss here the formation of select mortality tables. If we refer to the $O^{[M]}$ tables we find a mortality table in the following form:—

Age at Entry $[x]$	Years elapsed since Date of Assurance.								Age attained $x+n$
	0	1	2	etc.				n or more	
	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	etc.				l_{x+n}	

The most obvious way to form such a mortality table is to assume the same radix at each age at entry. Successive multiplication by the probabilities of living will then give us a complete table of mortality for each age at entry, since

$$l_{[x]} \times p_{[x]} = l_{[x]+1},\ l_{[x]+1} \times p_{[x]+1} = l_{[x]+2},$$

and generally where $k > n$

$$l_{[x]} \times p_{[x]} \times p_{[x]+1} \times \cdots \times p_{[x]+n-1} \times p_{x+n} \times p_{x+n+1} \times \cdots \times p_{x+k-1} = l_{x+k}$$

But by this method we should have an independent mortality table for each age at entry, and the extent of the monetary tables following thereon would be prohibitive. A better plan is, after forming the table for the first age at entry as above by working along the first line and down the column l_{x+n} to the limit of life, to form the remaining tables for the succeeding ages at entry by working backwards along their respective lines. Thus since

$$p_{[x+1]+n-1} = \frac{l_{x+n+1}}{l_{[x+1]+n-1}}$$

Therefore $$l_{[x+1]+n-1} = \frac{l_{x+n+1}}{p_{[x+1]+n-1}}$$

Also $$l_{[x+1]+n-2} = \frac{l_{[x+1]+n-1}}{p_{[x+1]+n-2}}$$ and so on.

But l_{x+n+1} is already calculated in the ultimate column, therefore $l_{[x+1]+n-1}$ may be found by dividing by $p_{[x+1]+n-1}$, and successively $l_{[x+1]+n-2}, \ldots l_{[x+1]}$. In the same way, commencing with l_{x+n+2}, we may work back to $l_{[x+2]}$, and so on for the other ages at entry.

The advantage of this method of formation is that only one set of "ultimate" monetary values is necessary.

29. Now out of $l_{[x]}$ select lives at age at entry x, l_{x+n} will be alive at the end of n years. But according to the method by which we have constructed our table l_{x+n} is also the number alive at the end of n years out of l_x mixed lives who were alive at age attained x. It follows therefore that the difference between $l_{[x]}$

and l_x is the number of damaged lives included in the l_x. Again, if from the deaths in each year of age amongst the l_x mixed lives we deduct the deaths in the corresponding years amongst the $l_{[x]}$ select lives we have the deaths in each year of age amongst the $(l_x - l_{[x]})$ damaged lives. The mortality experience is as follows:—

Age.	Survivors of			Number of Damaged Lives Dying in a Year.
	Mixed Lives.	Select Lives.	Damaged Lives.	
x	l_x	$l_{[x]}$	$l_x - l_{[x]}$	$d_x - d_{[x]}$
$x+1$	l_{x+1}	$l_{[x]+1}$	$l_{x+1} - l_{[x]+1}$	$d_{x+1} - d_{[x]+1}$
$x+2$	l_{x+2}	$l_{[x]+2}$	$l_{x+2} - l_{[x]+2}$	$d_{x+2} - d_{[x]+2}$
⋮	⋮	⋮	⋮	⋮
$x+n$	l_{x+n}	l_{x+n}	$l_{x+n} - l_{x+n} = 0$	0

It will be noticed that the effect of assuming that selection becomes unimportant after n years is that all the damaged lives die before the end of the n years.

30. To find the single premium required to permit of (x) effecting, n years hence without fresh medical examination, a whole-life assurance by annual premiums.

The annual premium to be paid by (x) is fixed at $P_{[x+n]}$, but considering that his medical examination takes place at the present time, the premium he should pay is $P_{[x]+n}$. The single premium to be paid now is therefore the present value of an annuity-due, of the difference between these two premiums, deferred n years, or

$$
\begin{aligned}
&{}_{n|}\mathrm{a}_{[x]}(P_{[x]+n} - P_{[x+n]})\\
&= \frac{D_{[x]+n}}{D_{[x]}}(P_{[x]+n} - P_{[x+n]})\,\mathrm{a}_{[x]+n}\\
&= \frac{(P_{[x]+n} - P_{[x+n]})N_{[x]+n}}{D_{[x]}}\\
&= (P_{[x]+n} - P_{[x+n]})(\mathrm{a}_{[x]} - \mathrm{a}_{[x]\overline{n|}})
\end{aligned}
$$

which is in the form most suitable for calculation.

For the annual premium for n years for such a benefit we should have

$$\frac{(P_{[x]+n} - P_{[x+n]})N_{[x]+n}}{N_{[x]} - N_{[x]+n}}$$

$$\text{or} \qquad \frac{(P_{[x]+n} - P_{[x+n]})(a_{[x]} - a_{[x]\overline{n}|})}{a_{[x]\overline{n}|}}$$

It is to be noted that this annual premium can only be accepted along with the annual premium for an ordinary policy running during the n years; otherwise by withdrawing at any time the assured would exercise against the office issuing the policy an option for which allowance has not been made in the calculation.

31. To find the annual premium for a short-term insurance upon the assumption that all the healthy lives withdraw at the end of the first year.

The necessity for taking such an option into consideration arises from the fact that the annual premium for a short-term insurance frequently diminishes with an increase in the age, the original date of termination of the contract remaining unchanged. Thus, in symbols, it may happen that

$$P^1_{x\overline{n}|} > P_{\overset{1}{x+1}:\overline{n-1}|} > P_{\overset{1}{x+2}:\overline{n-2}|} > \text{etc.} > P_{\overset{1}{x+n-1}:\overline{1}|}$$

The reason for this is that the decrease in the term of the insurance has a greater effect in reducing the premium than the increase in the age has in raising it. The following table (based on the $O^{[NM]}$ Table at $3\frac{1}{2}$ per cent.) illustrates the point.

Short-Term Insurance Premiums per unit assured.

Age.	Term.							Age.
	1.	2.	3.	4.	5.	6.	7.	
30	·00470	·00552	·00615	·00663	·00700	·00729	·00752	30
31	·00479	·00562	·00627	·00677	·00715	·00745	·00769	31
32	·00491	·00575	·00640	·00692	·00732	·00763	·00788	32
33	·00504	·00588	·00655	·00708	·00750	·00782	·00809	33
34	·00517	·00602	·00671	·00726	·00770	·00804	·00832	34
35	·00533	·00619	·00690	·00747	·00792	·00828	·00858	35
36	·00548	·00636	·00709	·00768	·00815	·00853	·00886	36
37	·00565	·00657	·00731	·00792	·00842	·00882	·00916	37
38	·00587	·00678	·00754	·00818	·00870	·00913	·00949	38
39	·00605	·00699	·00779	·00845	·00901	·00946	·00984	39
40	·00630	·00726	·00808	·00878	·00936	·00988	·01025	40

Here it will be seen that $P^{\,1}_{[80]:\overline{7}|} > P^{\,1}_{[81]:\overline{6}|} >$ etc. $> P^{\,1}_{[86]:\overline{1}|}$, and so on.

It follows, therefore, that an office issuing policies, say for seven years, runs an appreciable risk, in that all the lives which are still select at the end of the first year may drop their policies and effect new ones for the remaining six years at a lower rate. The value of this risk is ascertained in the problem before us.

If $l_{[x]}$ persons enter at age x, $d_{[x]}$ die within the first year, and $l_{[x+1]}$ withdraw at the end of the year. Therefore the number to enter on the second year is $l_{[x]+1} - l_{[x+1]}$. Out of these, $d_{[x]+1} - d_{[x+1]}$ die within the second year, leaving $l_{[x]+2} - l_{[x+1]+1}$ alive; $d_{[x]+2} - d_{[x+1]+1}$ die within the third year, leaving $l_{[x]+3} - l_{[x+1]+2}$ alive, and so on.

Therefore the benefit side

$$= \frac{vd_{[x]} + v^2(d_{[x]+1} - d_{[x+1]}) + v^3(d_{[x]+2} - d_{[x+1]+1}) + \cdots + v^n(d_{[x]+n-1} - d_{[x+1]+n-2})}{l_{[x]}}$$

$$= \frac{vd_{[x]} + v^2 d_{[x]+1} + v^3 d_{[x]+2} + \cdots + v^n d_{[x]+n-1}}{l_{[x]}}$$

$$- \frac{v^2 d_{[x+1]} + v^3 d_{[x+1]+1} + \cdots + v^n d_{[x+1]+n-2}}{l_{[x]}}$$

$$= \frac{M_{[x]} - M_{[x]+n}}{D_{[x]}} - \frac{M_{[x+1]} - M_{[x+1]+n-1}}{D_{[x]}}$$

And the payment side

$$= P'^{\,1}_{[x]\,\overline{n}|}$$

$$\times \left\{ \frac{l_{[x]} + v(l_{[x]+1} - l_{[x+1]}) + v^2(l_{[x]+2} - l_{[x+1]+1}) + \cdots + v^{n-1}(l_{[x]+n-1} - l_{[x+1]+n-2})}{l_{[x]}} \right\}$$

$$= P'^{\,1}_{[x]\,\overline{n}|}\left(\frac{l_{[x]} + vl_{[x]+1} + v^2 l_{[x]+2} + \cdots + v^{n-1} l_{[x]+n-1}}{l_{[x]}} \right.$$

$$\left. - \frac{vl_{[x+1]} + v^2 l_{[x+1]+1} + \cdots + v^{n-1} l_{[x+1]+n-2}}{l_{[x]}} \right)$$

$$= P'^{\,1}_{[x]\,\overline{n}|}\left(\frac{N_{[x]} - N_{[x]+n}}{D_{[x]}} - \frac{N_{[x+1]} - N_{[x+1]+n-1}}{D_{[x]}} \right)$$

Whence on equating and solving

$$P'^{\,1}_{[x]\,\overline{n}|} = \frac{(M_{[x]} - M_{[x]+n}) - (M_{[x+1]} - M_{[x+1]+n-1})}{(N_{[x]} - N_{[x]+n}) - (N_{[x+1]} - N_{[x+1]+n-1})}$$

If n be greater than the number of years during which the effect of selection is assumed to persist, this formula reduces to

$$P'^{\,1}_{[x]\,\overline{n}|} = \frac{M_{[x]} - M_{[x+1]}}{N_{[x]} - N_{[x+1]}}$$

32. To find the annual premium for a "Half Premium" Policy (see *Text Book*, Chapter XVI., Article 37), on the assumption that all the healthy lives withdraw at the end of t years, the half-premium term.

If $l_{[x]}$ persons enter at age x, $d_{[x]}$, $d_{[x]+1}$, . . . $d_{[x]+t-1}$, die in the first t years respectively, and at the end of the tth year $l_{[x+t]}$ withdraw, leaving to enter the $(t+1)$th year $l_{[x]+t} - l_{[x+t]}$, out of whom $d_{[x]+t} - d_{[x+t]}$ die in the $(t+1)$th year and $l_{[x]+t+1} - l_{[x+t]+1}$ enter the $(t+2)$th year, and so on.

$$\text{Benefit side} = \frac{1}{l_{[x]}}\Big\{vd_{[x]} + v^2d_{[x]+1} + \cdots + v^td_{[x]+t-1}$$

$$+ v^{t+1}(d_{[x]+t} - d_{[x+t]}) + v^{t+2}(d_{[x]+t+1} - d_{[x+t]+1}) + \cdots\Big\}$$

$$= \frac{M_{[x]} - M_{[x+t]}}{D_{[x]}}$$

$$\text{Payment side} = P\frac{1}{l_{[x]}}\Big[\,l_{[x]} + vl_{[x]+1} + \cdots + v^{t-1}\,l_{[x]+t-1}$$

$$+ 2\{v^t(l_{[x]+t} - l_{[x+t]}) + v^{t+1}(l_{[x]+t+1} - l_{[x+t]+1}) + \cdots\}\Big]$$

$$= P\frac{N_{[x]} + N_{[x]+t} - 2N_{[x+t]}}{D_{[x]}}.$$

Whence on equating and solving

$$P = \frac{M_{[x]} - M_{[x+t]}}{N_{[x]} + N_{[x]+t} - 2N_{[x+t]}}$$

33. To find the addition that must be made to the annual premium for a contingent insurance policy, payable in the event of (x) dying before (y), in order that (x) may, in the event of

surviving (y), have the option to effect a new whole-life assurance by annual premiums without fresh medical examination.

It will be convenient to assume that, should (x) survive (y), he will be alive at the end of the year of death of (y) when the option will fall to be exercised. Then in respect of the nth year the value of the option is

$$v^n{}_n p_{[x]}({}_{n-1}p_{[y]} - {}_n p_{[y]})(P_{[x]+n} - P_{[x+n]})a_{[x]+n}$$

To get at the complete value of the option this expression should be summed for each value of n from 1 onwards; and the annual option premium required will be found by dividing the result by $a_{[x][y]}$.

An approximate result would be obtained by taking the expectation of (y), say n years, and applying the formula for the problem discussed in paragraph 30, page 143.

34. To find the annual premium for a policy under which the sum assured, instead of being payable in one sum at death, is payable by instalments over a period of n years.

Here the benefit to be received at death of (x) is an annuity-due of $\frac{1}{n}$ per annum for n years certain, or $\frac{\text{a}_{\overline{n|}}}{n}$. Therefore if the annual premium for an assurance of 1 be P_x, that for an assurance of $\frac{\text{a}_{\overline{n|}}}{n}$ will be $P_x \frac{\text{a}_{\overline{n|}}}{n}$.

A further development of this policy consists in the guarantee that the instalments will be continued, even after the n years, during the lifetime of some nominated beneficiary. The value of this extended portion of the benefit is $\frac{1}{n}(a_y - a_{y:\overline{x(\overline{n|})}})$ and the premium for it should be made payable during the joint lifetime of (x) and (y). The reason for this is that on the death of (y), if before (x), the benefit is very much reduced and if the premium continue at the same rate as before, (x) may throw up the policy and insure at a lower premium if in good health, and thus exercise an option against the company. We therefore fix the annual premium at

$$P_x \frac{\text{a}_{\overline{n|}}}{n} + \frac{1}{n}\left(\frac{a_y - a_{y:\overline{x(\overline{n|})}}}{\text{a}_{xy}}\right)$$

with the agreement that, in the event of (y)'s dying before (x) it will be reduced to $P_x \dfrac{a_{\overline{n}|}}{n}$.

35. A cognate problem is to find the annual premium for an endowment assurance where the instalments are to commence at death or maturity and to be payable for a fixed period with continuance thereafter so long as the life assured survives. A definite number of payments is guaranteed, so that, in the event of death before maturity or within n years thereafter, the income would be payable for the minimum period agreed upon. On the principles above indicated the annual premium is

$$P_{x\overline{m}|}\frac{a_{\overline{n}|}}{n} + \frac{1}{n}\frac{{}_{m+n-1|}a_x}{a_{x\overline{m}|}}$$

36. To find the annual premium for a whole-life policy, it being a condition that the office retain the sum assured for n years after death of (x) and pay interest thereon for that period at rate j.

The office must settle upon the rate which it is to assume it will earn upon its investments, say i; and then the problem is simply to find the annual premium for a sum assured of $\{1+(j-i)a_{\overline{n}|(i)}\}$. The payments of the annuity here will obviously make up the rate of interest from i to j per unit as required. The annual premium will therefore be

$$P_x\{1+(j-i)a_{\overline{n}|(i)}\}$$

37. To find the annual premium for a double-endowment assurance, *i.e.*, a term assurance of 1 coupled with a pure endowment of 2.

$$\begin{aligned} P &= \frac{A^1_{x\overline{n}|} + 2A_{x\overline{n}|}^{\;\;1}}{1+a_{x:\overline{n-1}|}} \text{ or } \frac{A_{x\overline{n}|} + A_{x\overline{n}|}^{\;\;1}}{1+a_{x:\overline{n-1}|}} \\ &= \frac{M_x - M_{x+n} + 2D_{x+n}}{N_{x-1} - N_{x+n-1}} \\ &= P_{x\overline{n}|} + \frac{a_{x\overline{n}|} - a_{x:\overline{n-1}|}}{1+a_{x:\overline{n-1}|}} \end{aligned}$$

from which the premium may most conveniently be calculated.

A peculiarity of double-endowment assurances is that, unlike whole-life assurances and endowment assurances, the premium

generally decreases with an increase in the age, as may be observed from the rates in the following table :—

$O^{[M]}$ 3 *per cent. Double-Endowment Premiums*

Annual Premiums required for* £100 *payable in the event of death within the term, or* £200 *payable in the event of the term being survived.

Age.	Term.				
	10 years.	15 years.	20 years.	25 years.	30 years.
25	£16 17 3	£10 8 1	£7 4 7	£5 7 3	£4 3 0
30	16 16 11	10 7 9	7 4 3	5 6 11	4 2 7
35	16 16 5	10 7 3	7 3 9	5 6 5	4 2 3
40	16 15 8	10 6 6	7 3 0	5 5 11	4 2 2

It must not, however, be assumed that when the rate of mortality is increased the premium is in all cases diminished. Mr A. Levine has shown (*J. I. A.*, xxxiv. 514), that when the extra mortality, as compared with the normal, is small at first but steadily increasing, the normal premium is ample (within limits of course as to age at entry and term), for the reason that the increased risk under the term assurance portion of the contract is equalised by the diminished chance of receiving the double endowment portion. On the other hand, when the extra mortality, as compared with the normal, is great at first but constant, or very great at first but decreasing, the normal premium is no longer sufficient, the diminution in the premium for the double endowment now being not large enough to counterbalance the extra risk under the term assurance.

The following table exhibits the case of an increasing extra mortality, resulting in an increased term assurance premium, a decreased endowment premium, and finally a decreased total premium for the benefit.

Age at entry Twenty-five. Term of Endowment 20 years.

Benefit.	Extra Mortality Premium.	Normal Premium.	Increase (+) or Decrease (−) in Premium required.
Term Assurance for £100 . .	£1 15 4	£0 14 7	+ £1 0 9
Pure Endowment for £200 . .	5 6 11	6 10 0	− 1 3 1
Double Endowment for £100/200 .	7 2 3	7 4 7	− 0 2 4

38. To find the annual premium for a joint-life endowment assurance.

We have

$$P_{xy\overline{n}|} = \frac{1}{1+a_{xy:\overline{n-1}|}} - d$$

$$\text{and } a_{xy:\overline{n-1}|} = a_{xy} - \frac{D_{x+n-1:y+n-1}}{D_{xy}} a_{x+n-1:y+n-1}$$

$$= a_{xy} - \frac{D_{x+n-1}}{D_x}\frac{l_{y+n-1}}{l_y} a_{x+n-1:y+n-1}$$

Having found the value of $a_{xy:\overline{n-1}|}$ from this formula, which is perhaps the most convenient for the purpose, we may very easily obtain $P_{xy\overline{n}|}$.

A method of approximating to the value of the annual premium for a joint-life endowment assurance is suggested by Mr Lidstone (*J. I. A.*, xxxiii. 354), viz.,

$$P_{xy\overline{n}|} = P_{x\overline{n}|} + P_{y\overline{n}|} - P_{\overline{n}|}$$

where $P_{\overline{n}|}$ is the premium payable in advance which will repay 1 in n years certain. This formula gives values for $P_{xy\overline{n}|}$ which are generally a little too small, but with practice allowance for the difference may be made.

39. To find the annual premium required to provide a last-survivor endowment assurance on (x) and (y), payable at the end of n years or previous death; that is, an assurance payable (1) at the death of the survivor of (x) and (y), if that event takes place within n years, or (2) at the end of n years, if one or both be then alive.

$$\text{Benefit side} = A_{\overline{xy}\,\overline{n}|} = A_{x\overline{n}|} + A_{y\overline{n}|} - A_{xy\overline{n}|}$$

$$\text{Payment side} = P_{\overline{xy}\,\overline{n}|}(1 + a_{\overline{xy}:\overline{n-1}|})$$

$$= P_{\overline{xy}\,\overline{n}|}(1 + a_{x:\overline{n-1}|} + a_{y:\overline{n-1}|} - a_{xy:\overline{n-1}|})$$

$$\text{Hence } P_{\overline{xy}\,\overline{n}|} = \frac{A_{x\overline{n}|} + A_{y\overline{n}|} - A_{xy\overline{n}|}}{1 + a_{x:\overline{n-1}|} + a_{y:\overline{n-1}|} - a_{xy:\overline{n-1}|}}$$

$$\text{Again, since } A_{\overline{xy}\,\overline{n}|} = 1 - d(1 + a_{\overline{xy}:\overline{n-1}|})$$

Therefore $$P_{\overline{xy}\,\overline{n|}} = \frac{1 - d(1 + a_{\overline{xy}:\overline{n-1|}})}{1 + a_{\overline{xy}:\overline{n-1|}}}$$

$$= \frac{1}{1 + a_{\overline{xy}:\overline{n-1|}}} - d$$

from which it will be seen that the simplest way of obtaining $P_{\overline{xy}\,\overline{n|}}$ will be to enter conversion tables (see next chapter) with the value of $a_{\overline{xy}:\overline{n-1|}} = a_{x:\overline{n-1|}} + a_{y:\overline{n-1|}} - a_{xy:\overline{n-1|}}$.

40. To find the annual premium for an annuity to the last survivor of (x) and (y), deferred for n years.

The present value of this annuity is

$$_{n|}a_{\overline{xy}} = {}_{n|}a_x + {}_{n|}a_y - {}_{n|}a_{xy}$$

which we shall call the benefit side.

If the premiums are to continue till the annuity is entered upon, their present value is

$$P \times {}_{|n}\text{a}_{\overline{xy}} = P(\text{a}_{x\overline{n|}} + \text{a}_{y\overline{n|}} - \text{a}_{xy\overline{n|}})$$

which is called the payment side.

Equating the benefit side to the payment side and solving for P, we have

$$P = \frac{_{n|}a_x + {}_{n|}a_y - {}_{n|}a_{xy}}{\text{a}_{x\overline{n|}} + \text{a}_{y\overline{n|}} - \text{a}_{xy\overline{n|}}}$$

There is a certain amount of risk involved in making the premium payable till the annuity is entered upon; for should (x) die, say, in the $(t+1)$th year, the value of the benefit to (y) at the end of that year is $_{n-t-1|}a_{y+t+1}$ and he could purchase such an annuity under a new contract at an annual premium of $\dfrac{_{n-t-1|}a_{y+t+1}}{\text{a}_{y+t+1:\overline{n-t-1|}}}$. Now it is quite possible that this latter premium might be less than the premium payable under the original contract. It follows therefore that the office might be the loser in not receiving P, the premium quoted, throughout the whole status assumed in the calculation.

The alternative plan is to make the premium payable during the joint lives only, under which

$$P = \frac{{}_{n|}a_x + {}_{n|}a_y - {}_{n|}a_{xy}}{\mathbf{a}_{xy\overline{n}|}}$$

Here again a risk is involved, for one of the lives, say (x), may be dying when the contract is entered into, and the office would thus be granting to (y) an annuity deferred n years at a totally inadequate premium.

The latter plan is, however, probably the better, provided some satisfactory evidence as to the health of (x) and (y) is obtained.

41. To find the annual premium for a joint-life temporary assurance, *i.e.*, payable if either of the two lives (x) and (y) should die within n years.

The benefit side

$$= A_{\overset{1}{xy}:\overline{n}|} = \frac{vd_{xy} + v^2 d_{x+1:y+1} + \cdot\cdot\cdot + v^n d_{x+n-1:y+n-1}}{l_{xy}}$$

and the payment side

$$= P_{\overset{1}{xy}:\overline{n}|}\left(\frac{l_{xy} + vl_{x+1:y+1} + \cdot\cdot\cdot + v^{n-1} l_{x+n-1:y+n-1}}{l_{xy}}\right)$$

Equating these two expressions and solving we get

$$\begin{aligned} P_{\overset{1}{xy}:\overline{n}|} &= \frac{vd_{xy} + v^2 d_{x+1:y+1} + \cdot\cdot\cdot + v^n d_{x+n-1:y+n-1}}{l_{xy} + vl_{x+1:y+1} + \cdot\cdot\cdot + v^{n-1} l_{x+n-1:y+n-1}} \\ &= v - \frac{vl_{x+1:y+1} + v^2 l_{x+2:y+2} + \cdot\cdot\cdot + v^n l_{x+n:y+n}}{l_{xy} + vl_{x+1:y+1} + \cdot\cdot\cdot + v^{n-1} l_{x+n-1:y+n-1}} \\ &= v - \frac{a_{xy\overline{n}|}}{\mathbf{a}_{xy\overline{n}|}} \end{aligned}$$

It may be mentioned that the value of $P_{\overset{1}{xy}:\overline{n}|}$ is very nearly the equivalent of $(P^1_{x\overline{n}|} + P^1_{y\overline{n}|})$, the sum of the term assurance premiums for the lives singly, so long as n remains small. Thus by Dr Sprague's Tables at $3\frac{1}{2}$ per cent., $P_{\overset{1}{30:30}:\overline{5}|} = 1{\cdot}473$ per cent., whereas $2P^1_{30:\overline{5}|} = 1{\cdot}482$ per cent., a difference of only $\cdot009$

per cent. Other examples should be worked to illustrate this fact, which may be explained thus:—

$$P_{\overset{1}{\overline{xy}}:\overline{n}|} = \frac{A_{\overset{1}{\overline{xy}}:\overline{n}|}}{a_{xy\overline{n}|}}$$

$$= \frac{A^1_{x\overline{n}|} + A^1_{y\overline{n}|} - A_{\overset{1}{xy}\,\overline{n}|}}{a_{xy\overline{n}|}}$$

$$= \frac{A^1_{x\overline{n}|}}{a_{xy\overline{n}|}} + \frac{A^1_{y\overline{n}|}}{a_{xy\overline{n}|}} - \frac{A_{\overset{1}{xy}\,\overline{n}|}}{a_{xy\overline{n}|}}$$

Now if n be small the value of $A_{\overset{1}{xy}\,\overline{n}|}$ is also small, for it is improbable that both (x) and (y) will die within a few years, and accordingly the last term of this expression may be ignored. Again, if n be small the value of the term annuity on the joint lives will be very nearly equal to the term annuity on each of the lives, or in other words $a_{xy\overline{n}|} = a_{x\overline{n}|} = a_{y\overline{n}|}$ nearly. Accordingly

$$P_{\overset{1}{\overline{xy}}:\overline{n}|} = \frac{A^1_{x\overline{n}|}}{a_{x\overline{n}|}} + \frac{A^1_{y\overline{n}|}}{a_{y\overline{n}|}} \text{ very nearly}$$

$$= P^1_{x\overline{n}|} + P^1_{y\overline{n}|} \text{ very nearly.}$$

It is to be noted that the two adjustments are of opposite effect on the result, the ignoring of the term $\dfrac{A_{\overset{1}{xy}\,\overline{n}|}}{a_{xy\,\overline{n}|}}$ increasing the premium, and the substituting of $a_{x\overline{n}|}$ and $a_{y\overline{n}|}$ for $a_{xy\overline{n}|}$ reducing the premium.

EXAMPLES

1. Prove that D_x is always greater than M_x.

$$\begin{aligned}
D_x &= v^x l_x \\
&= v^x(d_x + d_{x+1} + d_{x+2} + \cdots) \\
&> v^x(vd_x + v^2 d_{x+1} + v^3 d_{x+2} + \cdots) \\
&\qquad \text{since } v < 1 \\
&> C_x + C_{x+1} + C_{x+2} + \cdots \\
&> M_x
\end{aligned}$$

Or again, since the present value of 1 payable at the end of the year of death of (x) is clearly less than 1, we have

$$A_x = \frac{M_x}{D_x} < 1$$

Hence

$$D_x > M_x$$

2. Express in terms of the D and N columns and the rate of discount, the annual premiums for

(1) An endowment assurance to mature in n years.

(2) A whole-life assurance, premiums limited to n payments.

Subtract the second from the first and give a verbal interpretation of the result.

The premium for (1) is $\dfrac{D_x}{N_{x-1} - N_{x+n-1}} - d$

and for (2) $\dfrac{D_x - dN_{x-1}}{N_{x-1} - N_{x+n-1}}$

The difference is $\dfrac{dN_{x+n-1}}{N_{x-1} - N_{x+n-1}}$, which is the annual premium required to provide an annuity-due during the lifetime of (x) after n years, consisting of d, the interest in advance on 1, payment of which in the case of (2) is deferred from the end of the nth year —as would happen under (1)—to the end of the year of death of (x). During the n years the benefits are identical.

3. Find the rate of interest, given

(a) $a_x = 13{\cdot}257$ and $A_x = {\cdot}19304$

(b) $a_x = 13{\cdot}164$ and $P_x = {\cdot}04147$

(c) $A_x = {\cdot}19414$ and $P_x = {\cdot}00927$

Approximately (a) 6 per cent.; (b) 3 per cent.; (c) 4 per cent.

4. Find the value of A_{x+1}, having given $P_x = {\cdot}01662$, $a_x = 17{\cdot}155$, and $p_x = {\cdot}99229$.

We have $\quad A_{x+1} = 1 - d(1+a_{x+1})$

$$= 1 - d\,\frac{a_x}{vp_x}$$

$$= 1 - \frac{ia_x}{p_x}$$

The only unknown quantity in this expression is i, which from the given values of P_x and a_x we ascertain to be practically equal to ·04. Substituting this value for i we get $A_{x+1} = {}$·30847.

5. Required the cost of a deferred annuity, of which the first payment is to be made at the end of four years, and which is then to continue for twenty years certain and thereafter for so long as a life presently aged x may live.

The first part of the annuity is an annuity-due for twenty years certain deferred four years, and the second is an annuity-due on (x) deferred twenty-four years. The cost is therefore

$$v^4(1+a_{\overline{19}|}) + {}_{24|}\mathrm{a}_x$$

$$= (a_{\overline{28}|} - a_{\overline{8}|}) + \frac{N_{x+23}}{D_x}.$$

6. Give an algebraical proof that

$$a_x = \Sigma v^n(1+i)\frac{(1+i)^n - 1}{i}\,{}_{n-1|}q_x - 1$$

$$a_x = \frac{1 - A_x}{d} - 1$$

$$= \frac{1+i}{i} - A_x\frac{1+i}{i} - 1$$

$$= \frac{1+i}{i} - \Sigma v^n\,{}_{n-1|}q_x\,\frac{1+i}{i} - 1$$

$$= \Sigma v^n(1+i)^n\,{}_{n-1|}q_x\,\frac{1+i}{i} - \Sigma v^n\,{}_{n-1|}q_x\,\frac{1+i}{i} - 1$$

$$= \Sigma v^n(1+i)\frac{(1+i)^n - 1}{i}\,{}_{n-1|}q_x - 1$$

7. Give the formula for a whole-of-life assurance on (x) by three payments, the first to be made immediately, the second to be half the amount of the first and to be made at the end of three years, and the third to be half the amount of the second and to be made at the end of seven years.

$$\text{The benefit side} = \frac{M_x}{D_x}$$

$$\text{The payment side} = P\,\frac{D_x + \frac{1}{2}D_{x+3} + \frac{1}{4}D_{x+7}}{D_x}$$

$$\text{whence} \quad P = \frac{M_x}{D_x + \frac{1}{2}D_{x+3} + \frac{1}{4}D_{x+7}}$$

8. Investigate a formula for the annual premium payable during life for an assurance on the life of (x), the sum assured not to be paid in any event for twenty years from the date of the policy.

The benefit divides itself into two parts. If (x) should die within twenty years the sum assured is payable at the end of that period and its value is $v^{20}(1 - {}_{20}p_x)$. The other part is an assurance on (x) deferred twenty years, ${}_{20|}A_x$. Therefore the benefit side is equal to

$$v^{20}(1 - {}_{20}p_x) + {}_{20|}A_x.$$

The payment side $= P(1 + a_x)$.

$$\text{Hence} \qquad P = \frac{v^{20}(1 - {}_{20}p_x) + {}_{20|}A_x}{1 + a_x}$$

9. X has an income of J per annum; he can insure his life at P_x per unit; and investments will yield i per unit after his decease. How much of his income must he spend in premiums, in order that his representatives after his death may enjoy a perpetual income derived from the policy, exactly equal to the balance? Assume that the income is payable at the beginning of each year.

Let S be the sum for which X must insure his life. Then SP_x is the amount of his income which he spends in premiums, the balance of his income being $(J - SP_x)$. The income (payable at the beginning of the year) which will be derived from the

proceeds of the policy will be Sd, d being the interest in advance corresponding to i. We now have the equation

$$\mathrm{J} - \mathrm{SP}_x = \mathrm{S}d$$

$$\text{whence} \quad \mathrm{S} = \frac{\mathrm{J}}{\mathrm{P}_x + d}$$

$$\text{and} \quad \mathrm{S\,P}_x = \frac{\mathrm{JP}_x}{\mathrm{P}_x + d}$$

10. In consideration of a yearly premium of $\dfrac{v^{20}}{\mathrm{a}_{x:\overline{20}|}}$, an assurance company offers a life aged x a policy securing a sum of 1 payable at the expiration of twenty years, if (x) be then alive, and a sum of S payable fifteen years after the end of the year of death of (x) if this event take place during the twenty years. Find the value of S.

The value of the premiums to be received is

$$\frac{v^{20}}{\mathrm{a}_{x:\overline{20}|}} \times \mathrm{a}_{x:\overline{20}|} = v^{20}$$

And the value of the benefit granted is

$$\frac{\mathrm{D}_{x+20} + \mathrm{S}v^{15}(\mathrm{M}_x - \mathrm{M}_{x+20})}{\mathrm{D}_x}$$

$$\text{Hence} \quad \mathrm{S}v^{15}(\mathrm{M}_x - \mathrm{M}_{x+20}) = v^{20}\mathrm{D}_x - \mathrm{D}_{x+20}$$

$$\text{and} \quad \mathrm{S} = \frac{v^5\mathrm{D}_x - (1+i)^{15}\mathrm{D}_{x+20}}{\mathrm{M}_x - \mathrm{M}_{x+20}}$$

11. Investigate the change in the value of q_x produced by assuming an increase in the rate of interest to represent an increase in the rate of mortality. Illustrate from the case where a_x is extracted from the 4 per cent. table and assumed to represent the 3 per cent. value of a table showing higher rates of mortality.

We must first obtain q_x in terms of values of a_x, and accordingly we have

$$a_x = vp_x(1 + a_{x+1})$$

$$p_x = \frac{(1+i)a_x}{1 + a_{x+1}}$$

$$q_x = 1 - p_x = 1 - \frac{(1+i)a_x}{1 + a_{x+1}}$$

Here, then, we get increased rates of mortality by taking a_x and a_{x+1} at a higher rate of interest while i remains the lower rate, and we may write $q'_x = 1 - \frac{(1+i)a'_x}{1+a'_{x+1}}$ where a'_x and a'_{x+1} are at rate j ($> i$).

To obtain the old rate of mortality from a similar formula we have

$$q_x = 1 - \frac{(1+j)a'_x}{1+a'_{x+1}}$$

$$\text{and} \quad q'_x - q_x = \left(1 - \frac{(1+i)a'_x}{1+a'_{x+1}}\right) - \left(1 - \frac{(1+j)a'_x}{1+a'_{x+1}}\right)$$

$$= \frac{(j-i)a'_x}{1+a'_{x+1}}$$

Taking the example of 3 per cent. and 4 per cent. we have the increase in q_x, *i.e.*, $q'_x - q_x = \frac{\cdot 01a'_x}{1+a'_{x+1}}$ where a'_x and a'_{x+1} are at 4 per cent.

12. Calculate from the values given below the net annual premiums at age thirty for the following policies :—

(*a*) Whole-life assurance, premiums payable throughout life.
(*b*) Whole-life assurance, premiums limited to ten payments.
(*c*) Ten years' temporary assurance.
(*d*) Ten years' pure endowment.
(*e*) Ten years' endowment assurance.
(*f*) Ten years' double-endowment assurance.

x	D_x	N_x	M_x
30	37879	805450	14419
31	36557	767571	14201
32	35272	731014	13980
33	34023	695742	13758
34	32808	661719	13535
35	31627	628911	13310
36	30480	597284	13083
37	29364	566804	12855
38	28279	537440	12626
39	27225	509161	12395
40	26201	481936	12164

(a) $\dfrac{M_{30}}{N_{30}} = \dfrac{14419}{805450} = \cdot 017902$

(b) $\dfrac{M_{30}}{N_{30}-N_{40}} = \dfrac{14419}{805450-481936} = \dfrac{14419}{323514} = \cdot 044570$

(c) $\dfrac{M_{30}-M_{40}}{N_{30}-N_{40}} = \dfrac{14419-12164}{805450-481936} = \dfrac{2255}{323514} = \cdot 006970$

(d) $\dfrac{D_{40}}{N_{30}-N_{40}} = \dfrac{26201}{323514} = \cdot 080989$

(e) $\dfrac{M_{30}-M_{40}+D_{40}}{N_{30}-N_{40}} = \cdot 006970 + \cdot 080989 = \cdot 087959$

(f) $\dfrac{M_{30}-M_{40}+2D_{40}}{N_{30}-N_{40}} = \cdot 087959 + \cdot 080989 = \cdot 168948$

13. What is the annual premium at 3 per cent. for a temporary insurance for three years on a life aged thirty? Given $l_{30}=92529$, $l_{31}=92079$, $l_{32}=91472$, $l_{33}=90763$.

$$P^{1}_{30:\overline{3}|} = v - \frac{vl_{31}+v^2l_{32}+v^3l_{33}}{l_{30}+vl_{31}+v^2l_{32}}$$
$$= \cdot 97087 - \cdot 96469$$
$$= \cdot 00618$$

14. Given P_x and A_{xx}, show how to find at rate of interest i the annual premiums for

(a) Joint-life Assurance on two lives aged x.
(b) Last-survivor Assurance on the same.

(a) $$P_{xx} = \frac{A_{xx}}{1+a_{xx}}$$
$$= \frac{dA_{xx}}{1-A_{xx}}$$

since $$a_{xx} = \frac{1-A_{xx}}{d} - 1$$

(b) $$P_{\overline{xx}} = \frac{A_{\overline{xx}}}{1+a_{\overline{xx}}}$$
$$= \frac{2A_x - A_{xx}}{1+2a_x - a_{xx}}$$

and $$A_x = \frac{P_x}{P_x+d}$$

$$a_x = \frac{1}{P_x+d} - 1$$

$$a_{xx} = \frac{1-A_{xx}}{d} - 1 \text{ as before.}$$

15. Find without using commutation columns an expression for the annual premium for an assurance payable only in the event of (x) and (y) both dying within n years.

$$P^{\,1}_{\overline{xy}\,\overline{n}|} = \frac{A^{\,1}_{\overline{xy}\,\overline{n}|}}{1+a_{\overline{xy}:\overline{n-1}|}}$$

$$= v - \frac{a_{\overline{xy}\,\overline{n}|}}{1+a_{\overline{xy}:\overline{n-1}|}}$$

where the values of $a_{\overline{xy}\,\overline{n}|}$ and $a_{\overline{xy}:\overline{n-1}|}$ are found from the formula

$$a_{\overline{xy}\,\overline{t}|} = a_{x\overline{t}|} + a_{y\overline{t}|} - a_{xy\overline{t}|}$$

$$= \frac{vl_{x+1}+v^2l_{x+2}+\cdot\cdot\cdot+v^t l_{x+t}}{l_x}$$

$$+ \frac{vl_{y+1}+v^2l_{y+2}+\cdot\cdot\cdot+v^t l_{y+t}}{l_y}$$

$$- \frac{vl_{x+1}\,l_{y+1}+v^2l_{x+2}\,l_{y+2}+\cdot\cdot\cdot+v^t l_{x+t}\,l_{y+t}}{l_x\,l_y}$$

16. Deduce the single and annual premiums for an assurance for ten years, payable as to one half at the first death and as to the other half at the second death of three lives (x), (y), and (z). Is there any practical objection to making the quotation, and if so, how would you propose to meet it?

The single premium is

$$\tfrac{1}{2}\left({}_{|10}A_{xyz} + {}_{|10}A_{\overline{xyz}}^{\,2}\right)$$

$$= \tfrac{1}{2}\left[\{v(1+a_{xyz:\overline{9}|}) - a_{xyz:\overline{10}|}\} + \{v(1+a_{\overline{xyz}:\overline{9}|}^{\,2}) - a_{\overline{xyz}:\overline{10}|}^{\,2}\}\right]$$

where $a_{\overline{xyz}:\overline{t}|}^{\,2} = a_{xy\overline{t}|} + a_{xz\overline{t}|} + a_{yz\overline{t}|} - 2a_{xyz\overline{t}|}$

In deducing the annual premium the above is the benefit side.

$$\text{Payment side} = \text{P}(1 + a_{\overline{xyz}:\overline{9}|}^{\;2})$$

if it is desired to make the premium level throughout the whole status. A certain risk, however, attaches to the issue of the policy on such a footing. If one of the lives, say (x), die early, the remainder of the benefit could be obtained by the survivors, (y) and (z), provided they are in good health, at a smaller premium than P as found from the above. In such a case the office would not receive the stipulated premium throughout the whole of the status assumed in the calculation. To get over the difficulty we may

(1) Make the premium payable only during the joint existence of all three lives, whence payment side $= \text{P}(1 + a_{xyz:\overline{9}|})$.

(2) A premium may be accepted which is to be reduced by half on the first death.

$$\text{Payment side} = \text{P}\{1 + \tfrac{1}{2}(a_{xyz:\overline{9}|} + a_{\overline{xyz}:\overline{9}|}^{\;2})\}.$$

(3) Probably the best way is to issue two policies, each for $\frac{1}{2}$, one of which will be payable on the first death, the premium being $\frac{1}{2}\,\text{P}_{\overset{1}{xyz}:\overline{10}|}$; and the other payable at the second death, with premium $\frac{1}{2}\,\text{P}_{|10}\text{A}_{\overline{xyz}}^{\;2}$.

17. Find the annual premium for an assurance of £100 payable as follows:—

(*a*) £50 at the death of the first of two lives and £50 at the death of the second, the premium to be reduced by one half from the date of first renewal after the death of the first life.

(*b*) £33, 6s. 8d. at the death of the first of three lives, the premium then to be reduced by one-third; £33, 6s. 8d. at the second death, with a similar reduction in the premium; and the remaining £33, 6s. 8d. on the death of the last survivor.

(*a*) The benefit is obviously $50(\text{A}_x + \text{A}_y)$.

The premium to be paid depends as to one-half on the life (x) irrespective of the life (y) and as to the other half on the life (y) irrespective of the life (x). The value of the payment side is therefore

$$\text{P}\{\tfrac{1}{2}(1 + a_x) + \tfrac{1}{2}(1 + a_y)\} = \text{P}\{1 + \tfrac{1}{2}(a_x + a_y)\}.$$

Or again, the payment side depends as to one-half on the joint lives of (x) and (y) and as to one-half on the life of the last survivor. That is,

$$\text{Payment side} = \text{P}(\tfrac{1}{2}\text{a}_{xy} + \tfrac{1}{2}\text{a}_{\overline{xy}}) = \text{P}\{1 + \tfrac{1}{2}(a_x + a_y)\}.$$

Hence $$\text{P} = 50\frac{\text{A}_x + \text{A}_y}{1 + \frac{1}{2}(a_x + a_y)}$$

(*b*) The premium required for this benefit, found in a manner similar to the above, is

$$33 \cdot \dot{3}\,\frac{\text{A}_x + \text{A}_y + \text{A}_z}{1 + \frac{1}{3}(a_x + a_y + a_z)}$$

The payment side may be expressed in either of the forms

$$\text{P}\{\tfrac{1}{3}(1 + a_x) + \tfrac{1}{3}(1 + a_y) + \tfrac{1}{3}(1 + a_z)\}$$

or $$\text{P}(\tfrac{1}{3}\text{a}_{xyz} + \tfrac{1}{3}\text{a}_{\overline{xyz}}^{\,2} + \tfrac{1}{3}\text{a}_{\overline{xyz}})$$

18. Find a formula for the annual premium, payable till the benefit is entered upon, for a deferred annuity of 1 to begin to run on either of two lives, presently aged twenty-five and thirty respectively, attaining age sixty. If (30) attain age sixty and (25) be also then alive the annuity will be payable thereafter during the joint lives and the life of the survivor.

$$\begin{aligned}\text{Benefit side} &= v^{30}{}_{30}p_{25:30}(a_{55} + a_{60} - a_{55:60}) \\ &\quad + v^{30}{}_{30}p_{30}(1 - {}_{30}p_{25})a_{60} + v^{35}{}_{35}p_{25}(1 - {}_{30}p_{30})a_{60}\end{aligned}$$

$$\text{Payment side} = \text{P}\{1 + {}_{|29}a_{\overline{25:30}} + v^{30}{}_{30}p_{25}(1 - {}_{30}p_{30})\,{}_{|5}\text{a}_{55}\}$$

$$\text{And P} = \frac{v^{30}{}_{30}p_{25:30}(a_{55} + a_{60} - a_{55:60}) + v^{30}{}_{30}p_{30}(1 - {}_{30}p_{25})a_{60} + v^{35}{}_{35}p_{25}(1 - {}_{30}p_{30})a_{60}}{1 + {}_{|29}a_{25} + {}_{|29}a_{30} - {}_{|29}a_{25:30} + v^{30}{}_{30}p_{25}(1 - {}_{30}p_{30})(1 + {}_{|4}a_{55})}$$

19. Find the annual premium for an assurance payable at the second death of the four lives (w), (x), (y), and (z).

$$\text{Benefit side} = \text{A}_{\overline{wxyz}}^{\,3}$$

$$\text{Payment side} = \text{P}_{\overline{wxyz}}^{\,3}(1 + a_{\overline{wxyz}}^{\,3})$$

Therefore equating

$$P_{\overline{wxyz}}^{\,3} = \frac{A_{\overline{wxyz}}^{\,3}}{1+a_{\overline{wxyz}}^{\,3}}$$

$$= \frac{1}{1+a_{\overline{wxyz}}^{\,3}} - d$$

where $a_{\overline{wxyz}}^{\,3} = a_{wxy} + a_{wxz} + a_{wyz} + a_{xyz} - 3a_{wxyz}$.

20. Find the single premium to assure a perpetuity of £100 in the event of A, who is aged thirty, dying within ten years, the first payment of the perpetuity to be due at the end of the year in which A dies. Interest is to be taken at 4 per cent. and the mortality is to be assumed to follow De Moivre's hypothesis. Given v^{10} at 4 per cent. = ·675564.

By De Moivre's hypothesis the number living at age thirty is $86 - 30 = 56$, and one dies every year. Therefore

$$A^{1}_{30:\overline{10|}} = \frac{v + v^2 + \cdot \cdot \cdot + v^{10}}{56}$$

$$= \frac{1 - v^{10}}{56i}$$

$$= \cdot 14484$$

The single premium for the perpetuity is

$$A^{1}_{30:\overline{10|}} \times 100 \times \frac{1+i}{i} = \cdot 14484 \times 100 \times 26.$$

$$= 376 \cdot 584$$

$$= £376,\ 11s.\ 8d.\ \text{nearly}.$$

21. On De Moivre's hypothesis as to the law of mortality, find the annual premium at age x to provide an endowment assurance payable at age $x+t$ or previous death.

$$P_{x\overline{t|}} = \frac{A_{x\overline{t|}}}{1+a_{x:\overline{t-1|}}}.$$

But on De Moivre's hypothesis where n represents the complement of life at age x, we have—

$$A_{x\overline{t}|} = \frac{v+v^2+\cdots+v^t+(n-t)v^t}{n}$$

$$= \frac{a_{\overline{t}|}+(n-t)v^t}{n}$$

Also $$a_{x:\overline{t-1}|} = \frac{v(n-1)+v^2(n-2)+\cdots+v^{t-1}(n-t+1)}{n}$$

$$= \frac{(n-1)a_{\overline{t-1}|} - a_{\overline{t-1}|\overline{2}|}}{n}$$

where $$a_{\overline{t-1}|\overline{2}|} = \frac{a_{\overline{t-1}|} - (t-1)v^{t-1}}{i}$$

Hence $$P_{x\overline{t}|} = \frac{a_{\overline{t}|}+(n-t)v^t}{n+(n-1)a_{\overline{t-1}|} - a_{\overline{t-1}|\overline{2}|}}$$

22. Find the value of an annuity to be payable until the survivor of three children, aged five, eleven, and thirteen respectively, attains majority.

$$a_{\overline{(5:\overline{16}|)(11:\overline{10}|)(13:\overline{8}|)}} = a_{5:\overline{16}|} + a_{11:\overline{10}|} + a_{13:\overline{8}|}$$

$$- a_{5:11:\overline{10}|} - a_{5:13:\overline{8}|} - a_{11:13:\overline{8}|}$$

$$+ a_{5:11:13:\overline{8}|}$$

23. Find the value of an assurance payable should three children, aged ten, twelve, and sixteen respectively, all die before attaining majority.

$$A_{\overline{(\overset{1}{10}:\overline{11}|)(\overset{1}{12}:\overline{9}|)(\overset{1}{16}:\overline{5}|)}} = A_{\overline{10:12:\overset{1}{16}}:\overline{5}|} + v^5(1-{}_5p_{16}){}_5p_{10:12}A_{\overline{15:\overset{1}{17}}:\overline{4}|}$$

$$+ v^5(1-{}_5p_{16})(1-{}_5p_{12}){}_5p_{10}A^{1}_{15:\overline{4}|} + v^5(1-{}_5p_{16})(1-{}_5p_{10}){}_5p_{12}A^{1}_{17:\overline{4}|}$$

$$+ v^9(1-{}_5p_{16})(1-{}_9p_{12}){}_9p_{10}A^{1}_{19:\overline{2}|}$$

24. How many damaged lives are there among the undermentioned 740,925 lives aged forty-five who have been insured for three years? and how many of them will die in each of the following five years?

Give a short sketch of the reasoning which leads to your figures.

Age x	Years elapsed since Date of Insurance. 0	1	2	3	4	5 or more.	Age x
	$l_{[x]}$	$l_{[x-1]+1}$	$l_{[x-2]+2}$	$l_{[x-3]+3}$	$l_{[x-4]+4}$	l_x	
45	730692	736363	739403	740925	741538	741700	45
46	720100	726015	729402	731220	731895	732100	46
47	709190	715320	718906	721033	721908	722100	47
48	698088	704292	708042	710325	711423	711700	48
49	686620	693058	696823	699220	700413	700800	49
50	674923	681445	685377	687728	688992	689400	50

The number alive at age forty-five of those who have been insured for three years is 740,925, but the number who are select at age forty-five is 730,692. Now both these numbers are reduced by mortality to the same figure at age fifty, namely 689,400. The surplus of 740,925 over 730,692 must therefore represent damaged lives who all die off in five years. The number of damaged lives among the 740,925 is thus 10,233 and the deaths amongst these in each year are shown as follows:—

Age. (1)	Number of Mixed Lives Surviving. (2)	Number of Lives Surviving out of those select at Age 45. (3)	Number of Damaged Lives Surviving. (2) - (3) (4)	Number of Damaged Lives Dying. (5)
45	740925	730692	10233	4353
46	731895	726015	5880	2686
47	722100	718906	3194	1819
48	711700	710325	1375	988
49	700800	700413	387	387
50	689400	689400	0	0

The difference at each age between the number surviving of those select at forty-five and the number surviving of those who at age forty-five had been insured three years shows the number of damaged lives surviving at each age, and the first differences of this column show the number dying in each of the five years.

25. Given a select mortality table showing $l_{[x]}$, $l_{[x-1]+1}$, $l_{[x-2]+2}$,

$l_{[x-3]+3}$, $l_{[x-4]+4}$, and l_x, express the probability of a select life aged x at entry being at the end of five years,

(*a*) In existence, irrespective of the state of his health then,
(*b*) In existence, and still a select life,
(*c*) In existence, and an unhealthy life.

In the form of tables described, selection is assumed to wear off in five years, therefore at the end of that time the number of $l_{[x]}$ persons select at age x who are still alive is merged in the ultimate table and is expressed by l_{x+5}, their health not being in consideration. But the number of select persons of age $x+5$ is by notation $l_{[x+5]}$, therefore the number of unhealthy is the remainder of the total l_{x+5}. Accordingly the probabilities required are

$$(a) \quad \frac{l_{x+5}}{l_{[x]}}$$

$$(b) \quad \frac{l_{[x+5]}}{l_{[x]}}$$

$$(c) \quad \frac{l_{x+5}-l_{[x+5]}}{l_{[x]}}$$

26. A life office secures every year K new assurers all aged x at entry. At the end of a quinquennium how many of the entrants during that time may be expected to be unhealthy.

Of $l_{[x]}$ persons who enter at age x there are alive at the end of five years $l_{[x]+5}$, of whom some are select and the others unhealthy. But the number of select lives of age $x+5$ is $l_{[x+5]}$. Therefore the number of unhealthy lives of that age is $l_{[x]+5}-l_{[x+5]}$. Similar expressions give the number of unhealthy lives at the end of four, three, two, and one years. Thus, if the number of entrants each year is K, and if we assume them all to enter at the beginning of the year, we get the number of unhealthy lives at the end of five years as

$$\frac{\mathrm{K}}{l_{[x]}}\Big\{(l_{[x]+5}-l_{[x+5]})+(l_{[x]+4}-l_{[x+4]})+(l_{[x]+3}-l_{[x+3]})$$
$$+(l_{[x]+2}-l_{[x+2]})+(l_{[x]+1}-l_{[x+1]})\Big\}$$

27. A person aged x wishes to be allowed to effect ten policies each for £1000 as follows:—

(1) At age	x	at the normal annual premium for that age.			
(2) ,,	$(x+1)$	,,	,,	,,	
(3) ,,	$(x+2)$	,,	,,	,,	
		etc.	etc.	etc.	
(10) ,,	$(x+9)$	,,	,,	,,	

It is required to find the single premium payable to provide for this option.

For the second of these policies the premium to be paid in absence of any arrangement would be $P_{[x]+1}$, whereas the premium arranged for is $P_{[x+1]}$. The value of the option on this one policy is therefore $(P_{[x]+1} - P_{[x+1]}) \dfrac{N_{[x]+1}}{D_{[x]}}$.

Similarly for the third policy the difference in premium to be allowed for is $(P_{[x]+2} - P_{[x+2]})$ and the value of this is $(P_{[x]+2} - P_{[x+2]}) \dfrac{N_{[x]+2}}{D_{[x]}}$. And so on for the other seven policies. The single premium to be paid for the option is therefore

$$1000\left\{(P_{[x]+1} - P_{[x+1]}) \frac{N_{[x]+1}}{D_{[x]}} + (P_{[x]+2} - P_{[x+2]}) \frac{N_{[x]+2}}{D_{[x]}} + \cdots \right.$$
$$\left. + (P_{[x]+9} - P_{[x+9]}) \frac{N_{[x]+9}}{D_{[x]}}\right\}$$

28. Calculate the following option premiums:—

(*a*) The single premium per cent. required to permit of (30) effecting at the end of five years a whole-life policy at the normal annual premium for his then age, without fresh medical examination. Use the $O^{[M]}$ Table at $3\frac{1}{2}$ per cent. interest.

(*b*) The yearly addition per cent. to the short-term insurance premium for seven years required to permit of (40) effecting a whole-life policy at the end of that period at the normal annual premium for his then age, without fresh medical examination. Use the $O^{[NM]}$ Table at 3 per cent. interest.

(*a*) Using the formula given on page 143, we have

$$\begin{aligned}&100(\mathrm{P}_{[80]+5}-\mathrm{P}_{[85]})(\mathrm{a}_{[80]}-\mathrm{a}_{[80]:\overline{5}|})\\ &=100(\cdot 02024-\cdot 01959)(19\cdot 793-4\cdot 633)\\ &=\cdot 985,\ \text{say } 19\text{s. } 8\text{d.}\end{aligned}$$

(*b*) Similarly for this option premium we have

$$\begin{aligned}&\frac{100(\mathrm{P}_{47}-\mathrm{P}_{[47]})(\mathrm{a}_{[40]}-\mathrm{a}_{[40]:\overline{7}|})}{\mathrm{a}_{[40]:\overline{7}|}}\\ &=\frac{(3\cdot 445-3\cdot 377)(18\cdot 102-6\cdot 250)}{6\cdot 250}\\ &=\cdot 129,\ \text{say } 2\text{s. } 7\text{d.}\end{aligned}$$

29. Find the annual office premium for a whole-life assurance to (x), the expenses being 8 per cent. of the first and subsequent gross premiums with further initial expenses of 2 per cent. on the sum assured and 5 per cent. on the first gross premium.

To get the value to the office of the payment side we must deduct from the value of the gross premium the value of all expenses. Thus, if P be the gross premium, the value of the payment side is

$$\mathrm{P}(1+a_x)-\cdot 08\mathrm{P}(1+a_x)-\cdot 02-\cdot 05\mathrm{P}$$

and this is equal to the value of the benefit, A_x.

Hence

$$\begin{aligned}\mathrm{P}(1+a_x)-\cdot 08\mathrm{P}(1+a_x)-\cdot 02-\cdot 05\mathrm{P}&=\mathrm{A}_x\\ \mathrm{P}\{\cdot 92(1+a_x)-\cdot 05\}&=\mathrm{A}_x+\cdot 02\\ \mathrm{P}&=\frac{\mathrm{A}_x+\cdot 02}{\cdot 92(1+a_x)-\cdot 05}\end{aligned}$$

30. Given the following office rates for immediate annuities on female single lives, aged fifty and sixty respectively, find at these ages the annuity which £500 will purchase, it being a condition that the annuity is to be payable during the joint lives and the lifetime of the survivor, but is to be reduced by one-half after the first death.

Age last Birthday.	Annuity which £100 will purchase.	Price of Annuity of £10.
50	£5 13 8	£175 19 1
60	7 5 10	137 2 10

The annuity required is the sum of two annuities of equal amount on the lives, and if 2P be the amount to be received during the joint lives we have

$$500 = P(a'_{50} + a'_{60})$$
$$= P(17{\cdot}5954 + 13{\cdot}7142)$$

$$\text{Hence}\quad P = \frac{500}{17{\cdot}5954 + 13{\cdot}7142}$$
$$= 15{\cdot}970, \text{ or say £15, 19s. 5d.}$$

£500 will therefore purchase an annuity of £31, 18s. 10d. during the joint lives, to be reduced to £15, 19s. 5d. on the first death.

31. Given tables of office premiums for endowment assurances and double-endowment assurances, show how to employ them to obtain the office premiums for the following benefits:—

(*a*) £100 payable on attaining a given age or at previous death, together with a guaranteed bonus of £33, 6s. 8d. payable only if the given age is attained.

(*b*) A similar benefit, but with a guaranteed bonus of £50.

(*a*) This is equivalent to a term assurance of £100 coupled with a pure endowment of £133, 6s. 8d., which may be split into an endowment assurance of £66, 13s. 4d. payable on attaining the given age or at previous death and a double-endowment assurance of £33, 6s. 8d. payable on death within the term and £66, 13s. 4d. on attaining the given age. Therefore if $P'_{\overline{xn}|}$ and $(DP)'_{\overline{xn}|}$ be the office premiums per £100 assured for endowment assurances and double-endowment assurances respectively, we obtain the required premium from the formula

$$\tfrac{2}{3}P'_{\overline{xn}|} + \tfrac{1}{3}(DP)'_{\overline{xn}|}.$$

(*b*) By a similar process of analysis we find the premium for the second benefit to be

$$\tfrac{1}{2}P'_{\overline{xn}|} + \tfrac{1}{2}(DP)'_{\overline{xn}|}.$$

32. From tables of office "Whole of Life" and "Limited Payment" premiums for each age at entry, show how to find the sum assured that could be given at age x for a single payment of S and a future annual payment of P.

Let A'_x and P'_x be the office single and annual premiums at age x per unit assured.

We must split the single payment to be made now into P and $(S-P)$ in order to put the payments of P on the basis of an annual premium.

Then since a single premium of A'_x insures 1, a single premium of $(S-P)$ insures $\dfrac{S-P}{A'_x}$.

Also since an annual premium of P'_x insures 1, an annual premium of P insures $\dfrac{P}{P'_x}$.

Therefore the whole amount insured by a single payment of S and a future annual payment of P is $\dfrac{S-P}{A'_x}+\dfrac{P}{P'_x}$.

33. A man aged forty next birthday desires to effect a policy payable at death for £5000. He proposes to make a first payment of £1000. Find the future premium to be charged annually, given tables as in the preceding question.

Let P be the future premium. Then by our formula above

$$\frac{1000-P}{A'_{40}}+\frac{P}{P'_{40}} = 5000$$

Hence
$$P\left(\frac{1}{P'_{40}}-\frac{1}{A'_{40}}\right) = 5000-\frac{1000}{A'_{40}}.$$

and
$$P = \frac{5000-\dfrac{1000}{A'_{40}}}{\dfrac{1}{P'_{40}}-\dfrac{1}{A'_{40}}}$$

34. Express in commutation form the annual premium for an endowment assurance to (x) payable in n years or at previous death, the premium for the first five years being only one-half of the premium for the remainder of the term. Find therefrom the premium for the first 5 years, and for the remaining 25 years, for a 30 years' endowment assurance on a life of 30. Interest 4 per cent. Given $N_{29}=502353{\cdot}5$; $N_{30}=474646{\cdot}5$; $N_{34}=376007{\cdot}4$; $N_{59}=58526{\cdot}9$; and $N_{60}=52931{\cdot}0$.

Here the Benefit side $= \dfrac{M_x - M_{x+n} + D_{x+n}}{D_x}$

And the Payment side $= P\left(\dfrac{N_{x-1} - N_{x+n-1}}{D_x} + \dfrac{N_{x+4} - N_{x+n-1}}{D_x}\right)$

Whence $$P = \frac{M_x - M_{x+n} + D_{x+n}}{N_{x-1} + N_{x+4} - 2N_{x+n-1}}.$$

Using now the figures given we have

$$P = \frac{M_{30} - M_{60} + D_{60}}{N_{29} + N_{34} - 2N_{59}}$$

But $M_{30} = vN_{29} - N_{30} = \dfrac{502353 \cdot 5}{1 \cdot 04} - 474646 \cdot 5 = 8385 \cdot 7$

$M_{60} = vN_{59} - N_{60} = \dfrac{58526 \cdot 9}{1 \cdot 04} - 52931 \cdot 0 = 3344 \cdot 9$

and $D_{60} = N_{59} - N_{60} = 58526 \cdot 9 - 52931 \cdot 0 = 5595 \cdot 9$

Therefore $$P = \frac{8385 \cdot 7 - 3344 \cdot 9 + 5595 \cdot 9}{502353 \cdot 5 + 376007 \cdot 4 - 117053 \cdot 8}$$
$$= \cdot 013972.$$

The premium for the first five years is accordingly ·013972 and thereafter ·027944.

35. Find by the $O^{[NM]}$ Table, with interest at $3\frac{1}{2}$ per cent. throughout, the annual premium per cent. required at age thirty to provide a debenture policy under which 5 per cent. interest payable half-yearly is to be provided on the sum assured for 15 years from the end of the year of death, at the end of which period the sum assured is to be payable.

As explained on page 148, the benefit at the end of the year of death is 1 plus an annuity-certain, for the period stipulated, of the excess of the guaranteed rate of interest over the rate assumed by the office. In this case the premium will therefore be

$$100\,P_{[30]}\left\{1 + \frac{\cdot 05 - \cdot 035}{2}\, a_{\overline{30|}\,(1\frac{3}{4}\%)}\right\}$$
$$= 1 \cdot 788(1 + \cdot 0075 \times 23 \cdot 18585)$$
$$= 2 \cdot 099, \text{ say } £2,\ 2s.$$

36. Find by the $O^{[NM]}$ Table, with interest at 3 per cent. throughout, the annual premium required at age thirty-five to provide an annuity-certain of £100 payable half-yearly for 20 years, the first payment to be made at the end of the year of death.

Modifying the formula given on page 147 to suit the case of a half-yearly annuity we have for this premium

$$\begin{aligned}P_{[35]} &\times 50 \times a_{\overline{40|}(1\frac{1}{2}\%)} \\ &= \cdot 02212 \times 50 \times 30\cdot 36458 \\ &= 33\cdot 583, \text{ say } £33, 11s.\ 8d.\end{aligned}$$

37. Find by Mr Lidstone's formula the annual premiums per cent. required for the following joint-life endowment assurances.

(*a*) Lives (30) and (35) for a term of 20 years on the basis of the $O^{[M]}$ Table with interest at $3\frac{1}{2}$ per cent.

(*b*) Lives (20) and (40) for a term of 30 years on the same basis with 3 per cent. interest.

(*a*) Following Mr Lidstone's formula we have

$$\begin{aligned}100(P_{[30]:\overline{20|}} &+ P_{[35]:\overline{20|}} - P_{\overline{20|}}) \\ &= 3\cdot 841 + 3\cdot 923 - 3\cdot 416 \\ &= 4\cdot 348, \text{ say } £4, 7s.\end{aligned}$$

(*b*)

$$\begin{aligned}100(P_{[20]:\overline{30|}} &+ P_{[40]:\overline{30|}} - P_{\overline{30|}}) \\ &= 2\cdot 460 + 2\cdot 947 - 2\cdot 041 \\ &= 3\cdot 366, \text{ say } £3, 7s.\ 4d.\end{aligned}$$

38. Write down formulas, with and without commutation symbols, for the annual premium for a joint-life term policy.

Calculate with the help of tables of logarithms the annual premium for a three-year term policy on the joint lives of A and B, each aged thirty-six next birthday, at 3 per cent. interest, having given

$\log l_{36} = 5\cdot 9097$, $\log l_{37} = 5\cdot 9076$, $\log l_{38} = 5\cdot 9042$, and $\log l_{39} = 5\cdot 9002$.

How would you approximate to such a premium in practice?

The annual premium in commutation symbols is

$$\frac{M_{xy} - M_{x+n:y+n}}{N_{x-1:y-1} - N_{x+n-1:y+n-1}}$$

from which we may pass to a formula without commutation symbols as follows :—

$$\frac{M_{xy} - M_{x+n\,:\,y+n}}{N_{x-1\,:\,y-1} - N_{x+n-1\,:\,y+n-1}}$$

$$= \frac{(vN_{x-1\,:\,y-1} - N_{xy}) - (vN_{x+n-1\,:\,y+n-1} - N_{x+n\,:\,y+n})}{N_{x-1\,:\,y-1} - N_{x+n-1\,:\,y+n-1}}$$

$$= v - \frac{N_{xy} - N_{x+n\,:\,y+n}}{N_{x-1\,:\,y-1} - N_{x+n-1\,:\,y+n-1}}$$

$$= v - \frac{vl_{x+1}\,l_{y+1} + v^2 l_{x+2}\,l_{y+2} + \cdots + v^n\,l_{x+n}\,l_{y+n}}{l_x\,l_y + vl_{x+1}\,l_{y+1} + \cdots + v^{n-1}\,l_{x+n-1}\,l_{y+n-1}}$$

Substituting the ages, etc., required for the second part of the question we have the premium desired as follows :—

$$v - \frac{vl_{37}\,l_{37} + v^2 l_{38}\,l_{38} + v^3 l_{39}\,l_{39}}{l_{36}\,l_{36} + vl_{37}\,l_{37} + v^2 l_{38}\,l_{38}}$$

At 3 per cent. $v = \cdot 97087$, also $\log v = \bar{1}\cdot 9872$, $\log v^2 = \bar{1}\cdot 9743$, and $\log v^3 = \bar{1}\cdot 9615$.

In dealing with the logarithms of the numbers living the characteristics may be ignored as they are the same in every case and only determine the position of the decimal place in the corresponding natural number.

Then $\log l_{36}\,l_{36} = 2 \log l_{36} = 2 \times \cdot 9097 = 1\cdot 8194 = \log 65\cdot 978$

$\log vl_{37}\,l_{37} = \log v + 2 \log l_{37} = \bar{1}\cdot 9872 + 1\cdot 8152 = 1\cdot 8024 = \log 63\cdot 445$

$\log v^2 l_{38}\,l_{38} = \log v^2 + 2 \log l_{38} = \bar{1}\cdot 9743 + 1\cdot 8084 = 1\cdot 7827 = \log 60\cdot 632$

$\log v^3 l_{39}\,l_{39} = \log v^3 + 2 \log l_{39} = \bar{1}\cdot 9615 + 1\cdot 8004 = 1\cdot 7619 = \log 57\cdot 796$

Therefore

$$v - \frac{vl_{37}\,l_{37} + v^2 l_{38}\,l_{38} + v^3 l_{39}\,l_{39}}{l_{36}\,l_{36} + vl_{37}\,l_{37} + v^2 l_{38}\,l_{38}}$$

$$= \cdot 97087 - \frac{63\cdot 445 + 60\cdot 632 + 57\cdot 796}{65\cdot 978 + 63\cdot 445 + 60\cdot 632}$$

$$= \cdot 97087 - \cdot 95695$$

$$= \cdot 01392.$$

In practice, as explained on page 152, we would take it that

$$P_{\overline{\overset{1}{|36:36|}}:\overline{3|}} = P^{1}_{36:\overline{3|}} + P^{1}_{36:\overline{3|}}$$

Using the figures given, we find $P^{1}_{36:\overline{8}|} = \cdot 00699$, and hence the approximate joint-life short-term premium is ·01398, which compares very favourably with the true value.

39. Find by the $O^{[NM]}$ Tables, using $3\frac{1}{2}$ per cent. interest, the annual premium for a joint-life short-term assurance for four years, the lives being aged thirty and forty.

Using the same formula as before

$$\begin{aligned}
P_{\overline{[30]:[40]}:\overline{4}|}^{\;1} &= v - \frac{v l_{[30]+1}\, l_{[40]+1} + \cdot \cdot \cdot + v^4 l_{[30]+4}\, l_{[40]+4}}{l_{[30]}\, l_{[40]} + \cdot \cdot \cdot + v^3 l_{[30]+3}\, l_{[40]+3}} \\
&= \cdot 96618 - \cdot 95086 \\
&= \cdot 01532.
\end{aligned}$$

The practical formula is in this case

$$\begin{aligned}
P_{\overline{[30]:[40]}:\overline{4}|}^{\;1} &= P^{1}_{30:\overline{4}|} + P^{1}_{40:\overline{4}|} \\
&= \cdot 00663 + \cdot 00878 \\
&= \cdot 01541.
\end{aligned}$$

40. Given $a_{[29][44]} = 13 \cdot 791$, $a_{[34][44]} = 13 \cdot 463$, $a_{[39][44]} = 13 \cdot 006$, and $a_{[44][44]} = 12 \cdot 391$, find the value of $a_{[37][44]}$ (1) by finite differences, (2) by central differences, stopping at second differences.

(1) For a finite-difference formula we have

$$a_{[x+n][y]} = a_{[x][y]} + \frac{n}{5}(a_{[x+5][y]} - a_{[x][y]}) + \frac{\frac{n}{5}\left(\frac{n}{5} - 1\right)}{\underline{|2}}(a_{[x+10][y]} - 2a_{[x+5][y]} + a_{[x][y]})$$

Hence

$$\begin{aligned}
a_{[37][44]} &= a_{[34][44]} + \tfrac{3}{5}(a_{[39][44]} - a_{[34][44]}) + \frac{\frac{3}{5}\left(-\frac{2}{5}\right)}{\underline{|2}}(a_{[44][44]} - 2a_{[39][44]} + a_{[34][44]}) \\
&= 13 \cdot 463 + \tfrac{3}{5}(13 \cdot 006 - 13 \cdot 463) - \tfrac{3}{25}(12 \cdot 391 - 26 \cdot 012 + 13 \cdot 463) \\
&= 13 \cdot 463 - \cdot 274 + \cdot 019 \\
&= 13 \cdot 208.
\end{aligned}$$

(2) The central-difference formula to be applied is

$$u_x = u_0 + x a_{+1} + \frac{x(x-1)}{\underline{|2}} b_0$$

Whence

$$\begin{aligned} a_{[37][44]} &= a_{[34][44]} + \tfrac{3}{5}(a_{[39][44]} - a_{[34][44]}) - \tfrac{3}{25}(a_{[39][44]} - 2a_{[34][44]} + a_{[29][44]}) \\ &= 13{\cdot}463 + \tfrac{3}{5}(13{\cdot}006 - 13{\cdot}463) - \tfrac{3}{25}(13{\cdot}006 - 26{\cdot}926 + 13{\cdot}791) \\ &= 13{\cdot}463 - {\cdot}274 + {\cdot}015 \\ &= 13{\cdot}204. \end{aligned}$$

41. Given $P_{40:25} = {\cdot}03625$, $P_{40:30} = {\cdot}0375$, $P_{45:25} = {\cdot}041$, and $P_{45:30} = {\cdot}04225$, find $P_{42:29}$.

$$\begin{aligned} P_{40:29} &= P_{40:25} + \tfrac{4}{5}(P_{40:30} - P_{40:25}) \\ &= {\cdot}03625 + \tfrac{4}{5}({\cdot}0375 - {\cdot}03625) \\ &= {\cdot}03625 + {\cdot}001 \\ &= {\cdot}03725. \\ P_{45:29} &= P_{45:25} + \tfrac{4}{5}(P_{45:30} - P_{45:25}) \\ &= {\cdot}041 + \tfrac{4}{5}({\cdot}04225 - {\cdot}041) \\ &= {\cdot}042. \\ P_{42:29} &= P_{40:29} + \tfrac{2}{5}(P_{45:29} - P_{40:29}) \\ &= {\cdot}03725 + \tfrac{2}{5}({\cdot}042 - {\cdot}03725) \\ &= {\cdot}03725 + {\cdot}0019 \\ &= {\cdot}03915. \end{aligned}$$

42. Complete the table of premiums between ages forty and forty-five, given the following:—

Age.	Premium.
40	£2 13 10
45	3 2 4
50	3 14 10
55	4 13 10

Denoting P_{40} as u_0, P_{45} as u_5, P_{50} as u_{10}, and P_{55} as u_{15}, we may proceed in either of two ways.

(*a*) We have

$$u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{\underline{|2}}\Delta^2 u_0 + \frac{x(x-1)(x-2)}{\underline{|3}}\Delta^3 u_0 + \ldots$$

and stopping at third differences the problem is to find

Δu_0, $\Delta^2 u_0$, and $\Delta^3 u_0$ from P_{40}, P_{45}, P_{50}, and P_{55}.

Now
$$\begin{aligned}
2{\cdot}692 &= u_0\\
3{\cdot}117 &= u_0 + 5\Delta u_0 + 10\Delta^2 u_0 + 10\Delta^3 u_0\\
3{\cdot}742 &= u_0 + 10\Delta u_0 + 45\Delta^2 u_0 + 120\Delta^3 u_0\\
4{\cdot}692 &= u_0 + 15\Delta u_0 + 105\Delta^2 u_0 + 455\Delta^3 u_0.
\end{aligned}$$

Differencing successively both sides
$$\begin{aligned}
{\cdot}425 &= 5\Delta u_0 + 10\Delta^2 u_0 + 10\Delta^3 u_0\\
{\cdot}625 &= 5\Delta u_0 + 35\Delta^2 u_0 + 110\Delta^3 u_0\\
{\cdot}950 &= 5\Delta u_0 + 60\Delta^2 u_0 + 335\Delta^3 u_0.
\end{aligned}$$

Again
$$\begin{aligned}
{\cdot}200 &= 25\Delta^2 u_0 + 100\Delta^3 u_0\\
{\cdot}325 &= 25\Delta^2 u_0 + 225\Delta^3 u_0.
\end{aligned}$$

And
$${\cdot}125 = 125\Delta^3 u_0.$$

whence we have
$$\begin{aligned}
\Delta^3 u_0 &= {\cdot}001\\
\Delta^2 u_0 &= {\cdot}004\\
\Delta u_0 &= {\cdot}075.
\end{aligned}$$

We make up the following scheme

Age.	Premium.	Δ	Δ^2	Δ^3
40	2·692	·075	·004	·001
41	2·767	·079	·005	
42	2·846	·084	·006	
43	2·930	·090	·007	
44	3·020	·097		
45	3·117			

(*b*) Alternatively we have
$$u_x = A + Bx + Cx^2 + Dx^3 + \cdot\ \cdot\ \cdot$$
and stopping at third powers of x the problem is to find B, C, and D from P_{40}, P_{45}, P_{50}, and P_{55}.
$$\begin{aligned}
2{\cdot}692 &= A\\
3{\cdot}117 &= A + 5B + 25C + 125D\\
3{\cdot}742 &= A + 10B + 100C + 1000D\\
4{\cdot}692 &= A + 15B + 225C + 3375D.
\end{aligned}$$

Differencing successively both sides of these equations,

$$\cdot 425 = 5B + 25C + 125D$$
$$\cdot 625 = 5B + 75C + 875D$$
$$\cdot 950 = 5B + 125C + 2375D$$

$$\cdot 200 = 50C + 750D$$
$$\cdot 325 = 50C + 1500D$$

$$\cdot 125 = 750D$$

whence

$$D = \cdot 0001\dot{6}$$
$$C = \cdot 0015$$
$$B = \cdot 07\dot{3}.$$

Then

$$P_{41} = 2\cdot 692 + \cdot 07\dot{3} + \cdot 0015 + \cdot 0001\dot{6} = 2\cdot 767$$
$$P_{42} = 2\cdot 692 + \cdot 14\dot{6} + \cdot 006 + \cdot 001\dot{3} = 2\cdot 846$$
$$P_{43} = 2\cdot 692 + \cdot 22 + \cdot 0135 + \cdot 0045 = 2\cdot 930$$
$$P_{44} = 2\cdot 692 + \cdot 29\dot{3} + \cdot 024 + \cdot 010\dot{6} = 3\cdot 020$$
$$P_{45} = 2\cdot 692 + \cdot 3\dot{6} + \cdot 0375 + \cdot 0208\dot{3} = 3\cdot 117$$

By either method we complete the table with the same values.

P_{40}	=	£2	13	10
P_{41}	=	2	15	4
P_{42}	=	2	16	11
P_{43}	=	2	18	7
P_{44}	=	3	0	5
P_{45}	=	3	2	4

CHAPTER VIII

Conversion Tables for Single and Annual Assurance Premiums

1. Conversion tables should be thoroughly understood, both in their construction and their use, and to these ends Rothery and Ryan's tables should be carefully examined.

For both single- and annual-premium conversion tables, they start with the initial value 1 for the annuity, and increase it by differences of ·01. Now we may draw up the following schedule for single-premium values :—

Annuity Value. (1)	Corresponding Assurance Value. (2)	Δ of Col. (2). (3)
1	$1-d(1+1)$	$-d\times{\cdot}01$
1·01	$1-d(1+1{\cdot}01)$	$-d\times{\cdot}01$
1·02	$1-d(1+1{\cdot}02)$	$-d\times{\cdot}01$
1·03	$1-d(1+1{\cdot}03)$	$-d\times{\cdot}01$
etc.	etc.	etc.

From this we see that the values in column (3) are constant and that therefore, the initial assurance value having been found, the successive values thereafter may be found by the continuous addition of $-d\times{\cdot}01$. The initial value must be found directly from $1-d(1+1)$. Verification may easily be made at periodical intervals. The correction for a third decimal place is found as shown in *Text Book*, Article 9.

2. Again, for annual premiums we make a preliminary investigation such as this :—

Annuity Value. (1)	Corresponding Annual Premium Value. (2)	Δ of Col. (2). (3)
1	$\frac{1}{1+1} - d$	$\frac{1}{1+1{\cdot}01} - \frac{1}{1+1}$
1·01	$\frac{1}{1+1{\cdot}01} - d$	$\frac{1}{1+1{\cdot}02} - \frac{1}{1+1{\cdot}01}$
1·02	$\frac{1}{1+1{\cdot}02} - d$	$\frac{1}{1+1{\cdot}03} - \frac{1}{1+1{\cdot}02}$
1·03	$\frac{1}{1+1{\cdot}03} - d$	$\frac{1}{1+1{\cdot}04} - \frac{1}{1+1{\cdot}03}$
etc.	etc.	etc.

Column (3) gives us the successive quantities to be added to the initial value in order to obtain the annual premiums which we require. We must therefore first proceed to find the values in column (3). The following schedule shows the process:—

Annuity Value $= a.$ (1)	$\frac{1}{1+a}$ (2)	Δ of Col. (2). (3)	P (4)
1	$\frac{1}{1+1}$	$\frac{1}{1+1{\cdot}01} - \frac{1}{1+1}$	$\frac{1}{1+1} - d$
1·01	$\frac{1}{1+1{\cdot}01}$	$\frac{1}{1+1{\cdot}02} - \frac{1}{1+1{\cdot}01}$	$\frac{1}{1+1{\cdot}01} - d$
1·02	$\frac{1}{1+1{\cdot}02}$	$\frac{1}{1+1{\cdot}03} - \frac{1}{1+1{\cdot}02}$	$\frac{1}{1+1{\cdot}02} - d$
etc.	etc.	etc.	etc.

We obtain the values in column (2) from a table of reciprocals and difference the results. These differences we then add successively to the initial value, $\frac{1}{1+1} - d$, to obtain the whole table of values at the rate involved in d. Column (3) consists of the differences both of column (2) and of column (4).

The differences in the value of P corresponding to a third place of decimals in the value of a must be found by interpolation which may be done quite simply.

As pointed out in *Text Book,* Article 16, the series of differences in column (3) is independent of the rate of interest and may be

used to form tables at each rate required, by successive addition to the proper initial value, $\frac{1}{1+1} - d$, where d varies with the rate.

Or again, after the first table at rate i is formed, that at any other rate, say j, may be formed by the constant addition of $d_{(i)} - d_{(j)}$ to each value, as may be seen from the following table:—

Annuity Value. (1)	Corresponding Annual Premium		(3) - (2) i.e., value to be added to Col. (2) to obtain Col. (3). (4)
	At Rate i. (2)	At Rate j. (3)	
1	$\frac{1}{1+1} - d_{(i)}$	$\frac{1}{1+1} - d_{(j)}$	$d_{(i)} - d_{(j)}$
1·01	$\frac{1}{1+1{\cdot}01} - d_{(i)}$	$\frac{1}{1+1{\cdot}01} - d_{(j)}$	$d_{(i)} - d_{(j)}$
1·02	$\frac{1}{1+1{\cdot}02} - d_{(i)}$	$\frac{1}{1+1{\cdot}02} - d_{(j)}$	$d_{(i)} - d_{(j)}$
etc.	etc.	etc.	etc.

where column (4) is constant.

3. The following is an alternative method of forming the conversion table from single to annual premiums.

$$P = \frac{dA}{1-A}$$

$$\text{Therefore } \frac{1}{P} = \frac{1-A}{dA}$$

$$= \frac{1}{d}\left(\frac{1}{A} - 1\right)$$

We must therefore make up a preliminary table of the reciprocals of the single-premium values, less unity. Then putting the value of $\frac{1}{d}$ on the fixed plate of the arithmometer and multiplying by each of the values in this preliminary table, we obtain a series of which we must again take the reciprocals to get the required values of P. This method does not fulfil the conditions of a continuous method. But good checks may easily be applied either by working backwards from the values of P to the values of A and comparing with the original values of A, or otherwise.

4. Conversion tables are of great value in the working out of premiums, whether single or annual, where it is frequently easier to obtain the annuity value than the premium. For example,

If we enter the Conversion Table with	We obtain in the Single-Premium Conversion Table	We obtain in the Annual-Premium Conversion Table
a_x	$1-d(1+a_x) \quad = A_x$	$\frac{1}{1+a_x}-d \quad = P_x$
$a_{x:\overline{n-1}\rvert}$	$1-d(1+a_{x:\overline{n-1}\rvert}) = A_{\overline{xn}\rvert}$	$\frac{1}{1+a_{x:\overline{n-1}\rvert}}-d = P_{\overline{xn}\rvert}$
a_{xy}	$1-d(1+a_{xy}) \quad = A_{xy}$	$\frac{1}{1+a_{xy}}-d \quad = P_{xy}$
$a_{xy:\overline{n-1}\rvert}$	$1-d(1+a_{xy:\overline{n-1}\rvert}) = A_{xy\overline{n}\rvert}$	$\frac{1}{1+a_{xy:\overline{n-1}\rvert}}-d = P_{xy\overline{n}\rvert}$
$a_{\overline{xy}}$	$1-d(1+a_{\overline{xy}}) \quad = A_{\overline{xy}}$	$\frac{1}{1+a_{\overline{xy}}}-d \quad = P_{\overline{xy}}$
$a_{\overline{xy}:\overline{n-1}\rvert}$	$1-d(1+a_{\overline{xy}:\overline{n-1}\rvert}) = A_{\overline{xy}\overline{n}\rvert}$	$\frac{1}{1+a_{\overline{xy}:\overline{n-1}\rvert}}-d = P_{\overline{xy}\overline{n}\rvert}$
$a_{\overline{n-1}\rvert}$	$1-d(1+a_{\overline{n-1}\rvert}) \quad = v^n$	$\frac{1}{1+a_{\overline{n-1}\rvert}}-d \quad = P_{\overline{n}\rvert}$

Thus when we speak of entering the single-premium table with $a_{x:\overline{n-1}\rvert}$ we mean that we give to a in the formula $1-d(1+a)$ the value $a_{x:\overline{n-1}\rvert}$, and the result from the table is $A_{\overline{xn}\rvert}$. If we entered with $a_{\overline{xn}\rvert}$ the result would be $A_{x:\overline{n+1}\rvert}$.

Also by entering the annual-premium table with $a_{\overline{n}\rvert}$ we give to a in the formula $\frac{1}{1+a}-d$ the value $a_{\overline{n}\rvert}$, and the result is

$$\frac{1}{1+a_{\overline{n}\rvert}}-d = \frac{v^{n+1}}{1+a_{\overline{n}\rvert}}$$
$$= P_{\overline{n+1}\rvert}$$

the annual premium for a sinking-fund assurance payable at the end of $(n+1)$ years.

5. To find the single and annual premiums for an assurance

payable on the death of the second of four lives (w), (x), (y), and (z).

$$\text{Now} \quad A_{\overline{wxyz}}^{\;3} = 1 - d(1 + a_{\overline{wxyz}}^{\;3})$$

that is, the assurance is not payable so long as 3 at least of the 4 lives survive, for 3 at least will cease to survive at the second death. The formula shows us that to obtain this assurance we enter the single-premium conversion table with $a_{\overline{wxyz}}^{\;3}$ which by formula (58) of *Text Book*, Chapter VII., is equal to

$$a_{wxy} + a_{wyz} + a_{wxz} + a_{xyz} - 3a_{wxyz}.$$

Similarly for the annual premium we enter the annual-premium conversion table with the same annuity, for

$$P_{\overline{wxyz}}^{\;3} = \frac{1}{1 + a_{\overline{wxyz}}^{\;3}} - d$$

6. When it happens that we are given the single- or annual-premium value, and the annuity value is required, we may obtain the latter by entering the ordinary conversion tables inversely, on the same principle as we obtain the natural number corresponding to any logarithm by entering ordinary log tables inversely.

Thus if we know P_x at any rate of interest we could find a_x from the formula

$$a_x = \frac{1}{P_x + d} - 1$$

but conversion tables enable us to find it at once by inspection.

Again let it be required to find $a_{xy\overline{n|}}$. We shall obtain the value required by entering inversely with $P_{xy:\overline{n+1|}}$ and by Lidstone's formula (described on page 150).

$$P_{xy:\overline{n+1|}} = P_{x:\overline{n+1|}} + P_{y:\overline{n+1|}} - P_{\overline{n+1|}} \text{ approximately.}$$

Also $P_{x:\overline{n+1|}}$, $P_{y:\overline{n+1|}}$, and $P_{\overline{n+1|}}$ may be found by entering the tables directly with $a_{x\overline{n|}}$, $a_{y\overline{n|}}$, and $a_{\overline{n|}}$ respectively.

Thus, find by the $O^{[M]}$ table at $3\frac{1}{2}$ per cent. the value of $a_{[30][40]:\overline{19|}}$.

We find from the $O^{[M]}$ table that $a_{[30]:\overline{19|}} = 12{\cdot}845$, $a_{[40]:\overline{19|}} = 12{\cdot}451$, also $a_{\overline{19|}} = 13{\cdot}710$ at $3\frac{1}{2}$ per cent., and on entering $3\frac{1}{2}$ per cent. conversion tables we have $P_{[30]:\overline{20|}} = {\cdot}03841$, $P_{[40]:\overline{20|}} = {\cdot}04052$, $P_{\overline{20|}} = {\cdot}03416$.

Therefore $P_{[80][40]:\overline{20}|} = \cdot 03841 + \cdot 04052 - \cdot 03416 = \cdot 04477$. And entering the same conversion table inversely we have $a_{[80][40]:\overline{19}|} = 11 \cdot 725$.

7. It should be clearly understood that conversion tables are merely a means of saving labour and further that they can be used to find the value of y, whatever y may be, so long as it can be expressed in the form, for single-premium tables,

$$1 - d(1+k)$$

and for annual-premium tables

$$\frac{1}{1+k} - d$$

where k is known and d is at a known rate of interest. It must be remarked that k does not need to be an annuity or at any rate recognisable as such.

$$\text{Thus let } A_{x\overline{t}|}^{\;1} = 1 - d(1+k)$$

$$\text{Then } d(1+k) = 1 - \frac{D_{x+t}}{D_x}$$

$$\text{and } \quad k = \frac{D_x - D_{x+t}}{dD_x} - 1$$

$$= \frac{vD_x - D_{x+t}}{dD_x}$$

Therefore if we enter the single-premium table with this function we obtain $A_{x\overline{t}|}^{\;1}$.

$$\text{Again, let } {}_tP_x = \frac{1}{1+k} - d$$

$$\text{Then } \quad \frac{1}{1+k} = \frac{M_x}{N_{x-1} - N_{x+t-1}} + d$$

$$= \frac{M_x + d(N_{x-1} - N_{x+t-1})}{N_{x-1} - N_{x+t-1}}$$

$$\text{and } \quad k = \frac{(N_{x-1} - N_{x+t-1}) - (M_x + dN_{x-1} - dN_{x+t-1})}{M_x + dN_{x-1} - dN_{x+t-1}}$$

$$= \frac{v(N_{x-1} - N_{x+t-1}) - M_x}{M_x + d(N_{x-1} - N_{x+t-1})}$$

Entering the annual-premium conversion table with the value of this expression we obtain ${}_tP_x$.

8. To form conversion tables for continuous functions we first sketch out the following scheme for single premiums :—

Annuity Value. (1)	Corresponding Assurance Value. (2)	Δ of Col. (2). (3)
1	$1-\delta$	$-\delta(\cdot01)$
1·01	$1-\delta(1\cdot01)$	$-\delta(\cdot01)$
1·02	$1-\delta(1\cdot02)$	$-\delta(\cdot01)$
etc.	etc.	etc.

Therefore starting with the initial value of $\overline{A}=1-\delta$, and by the continuous addition of $-\delta(\cdot01)$, we form the table at the rate of interest involved in δ.

Again, for annual premiums we have

Annuity Value. (1)	Corresponding Premium Value. (2)	Δ of Col. (2). (3)
1	$1-\delta$	$\frac{1}{1\cdot01}-1$
1·01	$\frac{1}{1\cdot01}-\delta$	$\frac{1}{1\cdot02}-\frac{1}{1\cdot01}$
1·02	$\frac{1}{1\cdot02}-\delta$	$\frac{1}{1\cdot03}-\frac{1}{1\cdot02}$
etc.	etc.	etc.

We must first then form a preliminary table of the reciprocals of the successive values of the annuity and find the differences of these reciprocals. Thereafter the successive addition of the differences to the initial value $1-\delta$ gives the table. As with ordinary tables, one series of differences is sufficient for the formation of tables at all rates of interest. That is to say, the constant addition of $\delta_{(i)}-\delta_{(j)}$ to the values at rate i will also yield us the table at rate j.

EXAMPLES

1. Verify by actual calculation the following values of A, which correspond to values of a advancing from 10 to 11 by differences of ·1, at $3\frac{3}{4}$ per cent. interest; and insert correct values in place of those which are incorrect.

a.	A.
10·0	·60241
·1	·59905
·2	·59508
·3	·59157
·4	·58775
·5	·58434
·6	·58072
·7	·57741
·8	·57349
·9	·56938
11·0	·56627

It will be found that the 2nd, 3rd, 5th, 8th, and 10th values are incorrect, the true values being ·59880, ·59518, ·58795, ·57711, and ·56988 respectively.

2. Check all the figures in the following table, using Rothery and Ryan's Conversion Tables to obtain the single and annual premiums.

Rate.	Basis.	a.	A.	P.
Whole-Life Assurance, age forty . . .	$O^{[NM]}2\frac{1}{2}\%$	18·280	·52976	·02748
Endowment Assurance, age thirty, payable at sixty	$O^{[M]}3\frac{1}{2}\%$	16·163	·41961	·02445
Leasehold Assurance, term 20 years . .	3%	14·324	·55367	·03613
Joint-Life Assurance, ages twenty-five and thirty-five . . .	$O^{[M]}3\%$	16·725	·48373	·02729
Absolute Reversion, *i.e.*, the present value of 1 payable at the death of (x), otherwise A_x, age seventy . .	$O^{[af]}4\%$	7·882	·65838	...

3. Given tables of joint-life annuities, the columns D and l for single lives, and conversion tables, show how, by means of these, you would arrive at a premium for a joint-life endowment assurance on (x) and (y) payable at the end of 20 years if both be alive, or at the first death before then, half premiums only to be payable for the first five years.

The benefit side $= A_{xy:\overline{20}|}$.

To obtain the value of this we must enter conversion tables with $a_{xy:\overline{19}|}$ which is equal to

$$a_{xy} - {}_{19|}a_{xy} = a_{xy} - v^{19}\,{}_{19}p_x \times {}_{19}p_y\, a_{x+19:y+19}$$
$$= a_{xy} - \frac{D_{x+19}}{D_x}\frac{l_{y+19}}{l_y} a_{x+19:y+19}$$

all the parts of which we obtain from the tables given.

The payment side $= P(1 + a_{xy:\overline{19}|} + {}_{5|}a_{xy:\overline{15}|})$

$a_{xy:\overline{19}|}$ we have found above and

$${}_{5|}a_{xy:\overline{15}|} = \frac{D_{x+5}}{D_x}\frac{l_{y+5}}{l_y}(1 + a_{x+5:y+5}) - \frac{D_{x+19}}{D_x}\frac{l_{y+19}}{l_y} a_{x+19:y+19}$$

Hence the value of P may be found.

4. Use the tables at the end of the Institute *Text Book* and tables of logarithms to find at 4 per cent. the single and annual premiums for a joint-life endowment assurance on two lives each aged thirty-seven, the sum assured to be payable at the end of 23 years or first death preceding.

Here we have

$$A_{37:37:\overline{23}|} = 1 - d(1 + a_{37:37:\overline{22}|})$$

$$\text{and } P_{37:37:\overline{23}|} = \frac{1}{1 + a_{37:37:\overline{22}|}} - d$$

$$\text{Now } a_{37:37:\overline{22}|} = a_{37:37} - \frac{N_{59:59}}{D_{37:37}}$$
$$= 13{\cdot}054 - 1{\cdot}524$$
$$= 11{\cdot}530$$

$\log N_{59:59} = 9{\cdot}40776$
$\log D_{37:37} = 9{\cdot}22491$
$\cdot18285$
$= \log\ 1{\cdot}524$

Entering the 4 per cent. single-premium conversion table with this value we get

Value corresponding to 11			= ·53846
Do.	·5	= ·01923	
Do.	·03	= ·00115	
Deduct			·02038
$A_{87:87:\overline{28}\rvert} =$			·51808

Entering the 4 per cent. annual-premium conversion table with 11·530 we get

Value corresponding to 11			= ·04487
Do.	·5	= ·00333	
Do.	·03	= ·00020	
Deduct			·00353
$P_{87:87:\overline{28}\rvert} =$			·04134

5. Given P, find the corresponding A by the use of the ordinary single- and annual-premium conversion tables.

Enter inversely with P the annual-premium conversion table and with the result thereby obtained enter directly the single-premium conversion table, which will give us the A required.

6. Show how to construct a conversion table from which A may be found directly by inspection, P being given.

Since $A = \dfrac{P}{P+d}$ we may proceed to form the table on the following system :—

Annual Premium Value. (1)	log (1). (2)	log{(1)+d} (3)	(2)−(3) = log A. (4)	log⁻¹(4) = A. (5)
P	$\log P$	$\log(P+d)$	$\log\dfrac{P}{P+d}$	$\dfrac{P}{P+d}$
$P+\Delta P$	$\log(P+\Delta P)$	$\log(P+\Delta P+d)$	$\log\dfrac{P+\Delta P}{P+\Delta P+d}$	$\dfrac{P+\Delta P}{P+\Delta P+d}$
$P+2\Delta P$	$\log(P+2\Delta P)$	$\log(P+2\Delta P+d)$	$\log\dfrac{P+2\Delta P}{P+2\Delta P+d}$	$\dfrac{P+2\Delta P}{P+2\Delta P+d}$
etc.	etc.	etc.	etc.	etc.

The work must be done in duplicate to ensure accuracy, as the method is not continuous.

Again since $\frac{1}{A} = \frac{P+d}{P} = 1 + \frac{d}{P}$, we may proceed by first drawing up a table of reciprocals of P, $P+\Delta P$, $P+2\Delta P$, etc.; then multiplying each of these by d on the arithmometer and adding 1 to each result we obtain a table of reciprocals of A from which the successive values of A may be found.

As this also is not a continuous method, the work must be done in duplicate; or checked by doing all the calculations in reverse order when we should obtain P, $P+\Delta P$, $P+2\Delta P$, etc.

7. If a single-premium continuous conversion table be entered inversely with $e^{-n\delta}$, what does the result obtained represent?

The equation upon which such a table is founded is

$$\bar{A} = 1 - \delta\bar{a}$$

And as we are to enter the table inversely we have

$$\bar{a} = \frac{1-\bar{A}}{\delta}$$

In the particular case before us, $\bar{A} = e^{-n\delta}$.

Therefore $\bar{a} = \frac{1-e^{-n\delta}}{\delta}$

which is the value of an annuity of 1 for n years payable momently with interest convertible momently. (See *Theory of Finance*, Chapter II., Formula (15).)

8. What would be the result of entering single- and annual-premium conversion tables, calculated for continuous functions, with $\bar{a}_{\overline{n|}}$; and what does $\frac{1}{\bar{a}_x} - \delta$ represent?

Entering with $\bar{a}_{\overline{n|}}$ we have

$$\begin{aligned}\bar{A} &= 1 - \delta\bar{a}_{\overline{n|}} \\ &= 1 - \delta\,\frac{1-e^{-n\delta}}{\delta} \\ &= e^{-n\delta}\end{aligned}$$

that is, the present value of 1 payable at the end of n years, interest being at i per annum convertible momently.

$$\text{Also } \bar{P} = \frac{1}{\bar{a}_{\overline{n|}}} - \delta$$
$$= \frac{\delta}{1-e^{-n\delta}} - \delta$$
$$= \frac{\delta}{1-e^{-n\delta}}\left\{1-(1-e^{-n\delta})\right\}$$
$$= \frac{e^{-n\delta}}{\bar{a}_{\overline{n|}}}$$

which is a year's premium payable by momently instalments, interest at rate i convertible momently, for a sinking-fund assurance due in n years.

$$\frac{1}{\bar{a}_x} - \delta = \frac{1-\delta\bar{a}_x}{\bar{a}_x}$$
$$= \frac{\bar{A}_x}{\bar{a}_x}$$

which is the premium payable per annum by momently instalments for a whole-life assurance payable at the moment of death. (See Chapters IX. and X.)

9. Find by means of Rothery and Ryan's Conversion Tables the single premium corresponding to annuity ·983, interest 4 per cent.

The practical difficulty here is that the tables start from unity for the annuity values, while the given annuity is less than unity.

$$\text{But} \qquad A = 1-d(1+a)$$
$$= 1-d(1+\overline{1+a})+d$$

Therefore enter the table with the value 1·983 and add d to the result.

Then the single premium corresponding to annuity 1·983 at 4 per cent.	=	·88526
d at 4 per cent.	=	·03846
Single premium corresponding to annuity ·983	=	·92372

The value of d might be obtained by taking the difference between the single premiums corresponding to annuities 1 and 2 respectively.

CHAPTER IX

Annuities and Premiums Payable Fractionally throughout the Year

1. Formula (1) of this chapter may be arrived at by the following method which is somewhat similar to that of *Text Book,* Article 3.

$$_{0|}\mathrm{a}_x = \mathrm{a}_x - 0$$

$$_{1|}\mathrm{a}_x = \mathrm{a}_x - 1$$

Therefore interpolating

$$_{\frac{k}{m}|}\mathrm{a}_x = \mathrm{a}_x - \frac{k}{m}$$

But
$$a_x^{(m)} = \frac{1}{m}\left(_{\frac{1}{m}|}\mathrm{a}_x + {}_{\frac{2}{m}|}\mathrm{a}_x + \cdot \cdot \cdot + {}_{\frac{m}{m}|}\mathrm{a}_x\right)$$
$$= \frac{1}{m}\left\{\left(\mathrm{a}_x - \frac{1}{m}\right) + \left(\mathrm{a}_x - \frac{2}{m}\right) + \cdot \cdot \cdot + \left(\mathrm{a}_x - \frac{m}{m}\right)\right\}$$
$$= \mathrm{a}_x - \frac{m+1}{2m}$$
$$= a_x + \frac{m-1}{2m}$$

Also
$$\mathrm{a}_x^{(m)} = a_x^{(m)} + \frac{1}{m}$$
$$= a_x + \frac{m+1}{2m}$$
$$= \mathrm{a}_x - \frac{m-1}{2m}$$

2. Formula (5) which applies to the case where $m = 2$ may be made general as follows :—

For each year which (x) completes, the amount to the end of the year of the m payments of $\frac{1}{m}$ each is

$$\frac{1}{m}\left\{(1+i)^{\frac{m-1}{m}}+(1+i)^{\frac{m-2}{m}}+\cdot\cdot\cdot+1\right\}$$

and the value of these payments for the whole life of (x) is

$$a_x\frac{1}{m}\left\{(1+i)^{\frac{m-1}{m}}+(1+i)^{\frac{m-2}{m}}+\cdot\cdot\cdot+1\right\}$$

Now in respect of the year of death, if (x) dies in the rth of the m periods $(r>1)$ the amount to the end of the year of the payments which he has received is

$$\frac{1}{m}(1+i)^{\frac{m-1}{m}}+\frac{1}{m}(1+i)^{\frac{m-2}{m}}+\cdot\cdot\cdot+\frac{1}{m}(1+i)^{\frac{m-r+1}{m}}$$

Taking the summation of this expression for every value of r from 2 to m, and dividing the result by $\frac{1}{m}$ since, on the assumption of a uniform distribution of deaths, death is equally probable in each of the m parts of the year, we have as the amount, at the end of the year of death, of the payments made during that year

$$\frac{1}{m}\left\{\frac{m-1}{m}(1+i)^{\frac{m-1}{m}}+\frac{m-2}{m}(1+i)^{\frac{m-2}{m}}+\cdot\cdot\cdot+\frac{1}{m}(1+i)^{\frac{1}{m}}\right\}$$

The value for the whole of life of the payments made during the year of death is therefore

$$\mathrm{A}_x\frac{1}{m}\left\{\frac{m-1}{m}(1+i)^{\frac{m-1}{m}}+\frac{m-2}{m}(1+i)^{\frac{m-2}{m}}+\cdot\cdot\cdot+\frac{1}{m}(1+i)^{\frac{1}{m}}\right\}$$

and we have

$$a_x^{(m)}=a_x\frac{1}{m}\left\{(1+i)^{\frac{m-1}{m}}+(1+i)^{\frac{m-2}{m}}+\cdot\cdot\cdot+1\right\}$$

$$+\mathrm{A}_x\frac{1}{m}\left\{\frac{m-1}{m}(1+i)^{\frac{m-1}{m}}+\frac{m-2}{m}(1+i)^{\frac{m-2}{m}}+\cdot\cdot\cdot+\frac{1}{m}(1+i)^{\frac{1}{m}}\right\}$$

Expanding the successive powers of $(1+i)$, but stopping at first powers of i in the expansions, we have

$$a_x^{(m)} = a_x\frac{1}{m}\left\{\left(1+\frac{m-1}{m}i\right)+\left(1+\frac{m-2}{m}i\right)+\ \cdot\ \cdot\ \cdot\ +1\right\}$$

$$+A_x\frac{1}{m}\left\{\frac{m-1}{m}\left(1+\frac{m-1}{m}i\right)+\frac{m-2}{m}\left(1+\frac{m-2}{m}i\right)+\cdots\right.$$

$$\left.+\frac{1}{m}\left(1+\frac{1}{m}i\right)\right\}$$

$$= a_x\frac{1}{m}\left(m+\frac{m-1}{2}i\right)+\frac{1-ia_x}{1+i}\frac{1}{m}\left\{\frac{m-1}{2}+\frac{(m-1)(2m-1)}{6m}i\right\}$$

$$= a_x+\frac{m-1}{2m}ia_x+\frac{m-1}{2m}+\frac{(m-1)(2m-1)}{6m^2}i-\frac{m-1}{2m}ia_x-\frac{m-1}{2m}i$$

$$\left(\text{taking } \frac{1}{1+i} = (1+i)^{-1} = 1-i \text{ approximately}\right)$$

$$= a_x+\frac{m-1}{2m}-\frac{m^2-1}{6m^2}i$$

If then in this formula we give to m the value 2 we have

$$a_x^{(2)} = a_x+\tfrac{1}{4}-\tfrac{1}{8}i$$

which is formula (5) of the *Text Book*.

This general formula can be rapidly applied in practice and on the whole gives very good results. If we calculate by it the annuities of *Text Book*, Article 28, we get the following values.

$$a_{30}^{(2)} = 20\cdot14127 \qquad a_{60}^{(2)} = 10\cdot46974$$

$$a_{30}^{(4)} = 20\cdot26533 \qquad a_{60}^{(4)} = 10\cdot59380$$

$$\bar{a}_{30} = 20\cdot39002 \qquad \bar{a}_{60} = 10\cdot71849$$

8. The argument of *Text Book*, Article 12, is very involved, and the following is an alternative method of obtaining formula (7), which is founded on the assumption of a uniform distribution of deaths.

$$a_x^{(m)} = \frac{1}{m}\frac{1}{l_x}\left(v^{\frac{1}{m}}l_{x+\frac{1}{m}} + v^{\frac{2}{m}}l_{x+\frac{2}{m}} + \cdots + v^{\frac{m-1}{m}}l_{x+\frac{m-1}{m}} + vl_{x+1}\right.$$
$$\left. + v^{1+\frac{1}{m}}l_{x+1+\frac{1}{m}} + v^{1+\frac{2}{m}}l_{x+1+\frac{2}{m}} + \cdots\right)$$
$$= \frac{1}{m}\frac{1}{l_x}\left\{v^{\frac{1}{m}}\left(l_{x+1} + \frac{m-1}{m}d_x\right) + v^{\frac{2}{m}}\left(l_{x+1} + \frac{m-2}{m}d_x\right) + \cdots\right.$$
$$+ v^{\frac{m-1}{m}}\left(l_{x+1} + \frac{1}{m}d_x\right) + vl_{x+1} + v^{1+\frac{1}{m}}\left(l_{x+2} + \frac{m-1}{m}d_{x+1}\right)$$
$$\left. + v^{1+\frac{2}{m}}\left(l_{x+2} + \frac{m-2}{m}d_{x+1}\right) + \cdots\right\}$$
$$= \frac{vl_{x+1} + v^2l_{x+2} + \cdots}{l_x} \times \frac{1}{m}\left\{(1+i)^{\frac{m-1}{m}} + (1+i)^{\frac{m-2}{m}} + \cdots + 1\right\}$$
$$+ \frac{vd_x + v^2d_{x+1} + \cdots}{l_x} \times \frac{1}{m^2}\left\{(m-1)(1+i)^{\frac{m-1}{m}} + (m-2)(1+i)^{\frac{m-2}{m}} + \cdots\right.$$
$$\left. + (1+i)^{\frac{1}{m}}\right\}$$
$$= a_x \times \frac{i}{m\{(1+i)^{\frac{1}{m}} - 1\}} + \mathrm{A}_x \times \frac{m(1+i)\{(1+i)^{\frac{1}{m}} - 1\} - i(1+i)^{\frac{1}{m}}}{m^2\{(1+i)^{\frac{1}{m}} - 1\}^2}$$

The sum of the expression which is the coefficient of A_x may be found as follows :—

Let $$\Sigma = (m-1)(1+i)^{\frac{m-1}{m}} + (m-2)(1+i)^{\frac{m-2}{m}} + \cdots + (1+i)^{\frac{1}{m}}$$
$$(1+i)^{\frac{1}{m}}\Sigma = (m-1)(1+i) + (m-2)(1+i)^{\frac{m-1}{m}} + \cdots + (1+i)^{\frac{2}{m}}$$
$$\Sigma\{(1+i)^{\frac{1}{m}} - 1\} = m(1+i) - \{(1+i) + (1+i)^{\frac{m-1}{m}} + \cdots + (1+i)^{\frac{1}{m}}\}$$
$$= m(1+i) - (1+i)^{\frac{1}{m}}\frac{i}{(1+i)^{\frac{1}{m}} - 1}$$

And $$\Sigma = \frac{m(1+i)\{(1+i)^{\frac{1}{m}} - 1\} - i(1+i)^{\frac{1}{m}}}{\{(1+i)^{\frac{1}{m}} - 1\}^2}$$

4. To find $a^{(m)}_{x\overline{t|}}$.

$$
\begin{aligned}
a^{(m)}_{x\overline{t|}} &= a^{(m)}_x - \frac{D_{x+t}}{D_x}a^{(m)}_{x+t}\\
&= a_x - \frac{m-1}{2m} - \frac{D_{x+t}}{D_x}\left(a_{x+t} - \frac{m-1}{2m}\right)\\
&= a_x - \frac{D_{x+t}}{D_x}a_{x+t} - \frac{m-1}{2m}\left(1 - \frac{D_{x+t}}{D_x}\right)\\
&= a_{x\overline{t|}} - \frac{m-1}{2m}\left(1 - \frac{D_{x+t}}{D_x}\right)
\end{aligned}
$$

5. To find $_{\frac{1}{t}|}a_x$, that is, the yearly annuity whose payments are made at the end of $\frac{1}{t}$, $1 + \frac{1}{t}$, $2 + \frac{1}{t}$, etc., years respectively. It is quite distinct from $_{\frac{1}{t}|}a^{(t)}_x$, the annuity payable t times a year in instalments of $\frac{1}{t}$ each at the end of $\frac{1}{t}$, $\frac{2}{t}$, $\frac{3}{t}$, etc., of a year respectively, which is the same as $a^{(t)}_x$.

$$
\begin{aligned}
{}_{\frac{1}{t}|}a_x &= \frac{1}{l_x}\left(v^{\frac{1}{t}}l_{x+\frac{1}{t}} + v^{1+\frac{1}{t}}l_{x+1+\frac{1}{t}} + v^{2+\frac{1}{t}}l_{x+2+\frac{1}{t}} + \cdots\right)\\
&= \frac{1}{D_x}\left(D_{x+\frac{1}{t}} + D_{x+1+\frac{1}{t}} + D_{x+2+\frac{1}{t}} + \cdots\right)
\end{aligned}
$$

Now

$$
D_{x+\frac{1}{t}} = D_x + \frac{1}{t}\Delta D_x + \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)}{\underline{|2}}\Delta^2 D_x + \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)\left(\frac{1}{t}-2\right)}{\underline{|3}}\Delta^3 D_x + \cdots
$$

$$
D_{x+1+\frac{1}{t}} = D_{x+1} + \frac{1}{t}\Delta D_{x+1} + \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)}{\underline{|2}}\Delta^2 D_{x+1} + \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)\left(\frac{1}{t}-2\right)}{\underline{|3}}\Delta^3 D_{x+1} + \cdots
$$

etc. etc. etc.

And summing the portion within brackets, we have

$$
N_{x-1} - \frac{1}{t}D_x - \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)}{2}\Delta D_x - \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)\left(\frac{1}{t}-2\right)}{6}\Delta^2 D_x - \cdots
$$

Therefore

$$\underset{\frac{1}{t}}{|}a_x = \frac{1}{D_x}\left\{N_{x-1} - \frac{1}{t}D_x - \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)}{2}\Delta D_x - \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)\left(\frac{1}{t}-2\right)}{6}\Delta^2 D_x - \cdots\right\}$$

$$= a_x - \frac{1}{t} - \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)}{2}\frac{\Delta D_x}{D_x} - \frac{\frac{1}{t}\left(\frac{1}{t}-1\right)\left(\frac{1}{t}-2\right)}{6}\frac{\Delta^2 D_x}{D_x} - \cdots$$

$$= a_x + \frac{t-1}{t} + \frac{t-1}{2t^2}\frac{\Delta D_x}{D_x} - \frac{(t-1)(2t-1)}{6t^3}\frac{\Delta^2 D_x}{D_x} + \cdots$$

which is *Text Book* formula (27).

6. To find $\underset{\frac{1}{t}}{|}a_{\overline{xn}|}$ we must sum only the first n terms of the expression in brackets and we have accordingly

$$\underset{\frac{1}{t}}{|}a_{\overline{xn}|} = a_{\overline{xn}|} - \frac{1}{t}\left(\frac{D_x - D_{x+n}}{D_x}\right) \text{ approximately}$$

$$= \frac{1}{t}\left(ta_{\overline{xn}|} - 1 + a_{\overline{xn}|} - a_{x:\overline{n-1}|}\right)$$

$$= \frac{1}{t}\left\{(t-1)a_{\overline{xn}|} + a_{\overline{xn}|}\right\}$$

7. To find $\underset{\frac{1}{t}}{|}a_x^{(m)}$.

$$\underset{\frac{1}{t}}{|}a_x^{(m)} = \frac{1}{m}\frac{1}{l_x}\left(v^{\frac{1}{t}}l_{x+\frac{1}{t}} + v^{\frac{1}{t}+\frac{1}{m}}l_{x+\frac{1}{t}+\frac{1}{m}} + v^{\frac{1}{t}+\frac{2}{m}}l_{x+\frac{1}{t}+\frac{2}{m}} + \cdots\right)$$

$$= \frac{1}{m}\frac{1}{D_x}\left(D_{x+\frac{1}{t}} + D_{x+\frac{1}{t}+\frac{1}{m}} + D_{x+\frac{1}{t}+\frac{2}{m}} + \cdots\right)$$

$$= \frac{1}{m}\frac{1}{D_x}\left\{(D_x + D_{x+\frac{1}{m}} + D_{x+\frac{2}{m}} + \cdots) - \frac{m}{t}D_x - \frac{\frac{m}{t}\left(\frac{m}{t}-1\right)}{2}\Delta D_x - \cdots\right\}$$

(where $\Delta D_x = D_{x+\frac{1}{m}} - D_x$)

$$= a_x^{(m)} - \frac{1}{t} + \frac{t-m}{2t^2}\frac{\Delta D_x}{D_x} - \cdots$$

8. To find ${}_{\frac{1}{t}|}\mathrm{a}^{(m)}_{\overline{xn}|}$.

$$
\begin{aligned}
{}_{\frac{1}{t}|}\mathrm{a}^{(m)}_{\overline{xn}|} &= \mathrm{a}^{(m)}_{x} - \frac{1}{t} - \frac{\mathrm{D}_{x+n}}{\mathrm{D}_x}\left(\mathrm{a}^{(m)}_{x+n} - \frac{1}{t}\right) \text{ approximately} \\
&= \mathrm{a}^{(m)}_{\overline{xn}|} - \frac{1}{t}\left(\frac{\mathrm{D}_x - \mathrm{D}_{x+n}}{\mathrm{D}_x}\right) \\
&= \frac{1}{t}\left\{t\mathrm{a}^{(m)}_{\overline{xn}|} - 1 + m\left(a^{(m)}_{\overline{xn}|} - a^{(m)}_{\overline{x:n-\frac{1}{m}}|}\right)\right\} \\
&= \frac{1}{t}\left\{t\mathrm{a}^{(m)}_{\overline{xn}|} - m\left(\frac{1}{m} + a^{(m)}_{\overline{x:n-\frac{1}{m}}|}\right) + ma^{(m)}_{\overline{xn}|}\right\} \\
&= \frac{1}{t}\left\{(t-m)\mathrm{a}^{(m)}_{\overline{xn}|} + ma^{(m)}_{\overline{xn}|}\right\}
\end{aligned}
$$

9. To find $\mathrm{P}^{(m)}_x$.

$$
\begin{aligned}
\mathrm{a}^{(m)}_x &= \mathrm{a}_x - \frac{m-1}{2m} \\
&= \mathrm{a}_x\left(1 - \frac{m-1}{2m}\frac{1}{\mathrm{a}_x}\right) \\
&= \mathrm{a}_x\left\{1 - \frac{m-1}{2m}(\mathrm{P}_x + d)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\text{Now}\quad \mathrm{P}^{(m)}_x &= \frac{\mathrm{A}_x}{\mathrm{a}^{(m)}_x} \\
&= \frac{\mathrm{A}_x}{\mathrm{a}_x\left\{1 - \frac{m-1}{2m}(\mathrm{P}_x + d)\right\}} \\
&= \frac{\mathrm{P}_x}{1 - \frac{m-1}{2m}(\mathrm{P}_x + d)}
\end{aligned}
$$

Again, multiplying each side by $\left\{1 - \frac{m-1}{2m}(\mathrm{P}_x + d)\right\}$, we have

$$\mathrm{P}^{(m)}_x - \frac{m-1}{2m}\mathrm{P}^{(m)}_x(\mathrm{P}_x + d) = \mathrm{P}_x$$

$$\text{and}\quad \mathrm{P}^{(m)}_x = \mathrm{P}_x + \frac{m-1}{2m}\mathrm{P}^{(m)}_x(\mathrm{P}_x + d)$$

from which we see that the addition to P_x to obtain $P_x^{(m)}$ mentioned in *Text Book*, Article 42, is for loss of premium $\frac{m-1}{2m} P_x^{(m)} P_x$ and for loss of interest $\frac{m-1}{2m} P_x^{(m)} d$.

The addition for loss of premium is found as follows:—The chance of the second instalment of the *m*thly premium not being received is $\frac{1}{m}$, of the third $\frac{2}{m}$, etc., of the *m*th $\frac{m-1}{m}$, on the assumption of uniform distribution of deaths. Therefore the whole premium lost in the year of death is

$$\frac{P_x^{(m)}}{m}\left(\frac{1}{m}+\frac{2}{m}+\cdot\cdot\cdot+\frac{m-1}{m}\right) = \frac{P_x^{(m)}}{m}\frac{m-1}{2};$$

and the annual premium required to insure against this loss is $\frac{m-1}{2m} P_x^{(m)} P_x$ as above.

The loss of interest on the second instalment being postponed $\frac{1}{m}$ of a year is $\frac{P_x^{(m)}}{m}\frac{d}{m}$; on the third being postponed $\frac{2}{m}$ of a year $\frac{P_x^{(m)}}{m}\frac{2d}{m}$, etc; on the *m*th being postponed $\frac{m-1}{m}$ of a year $\frac{P_x^{(m)}}{m}\frac{m-1}{m}d$. Therefore the loss of interest in accepting an *m*thly premium is

$$\frac{P_x^{(m)}}{m}\frac{d}{m}\left\{1+2+\cdot\cdot\cdot+(m-1)\right\} = \frac{P_x^{(m)}}{m}\frac{m-1}{2}d$$

$$= \frac{m-1}{2m} P_x^{(m)} d \text{ as above.}$$

10. To find $P_{x\bar{t}|}^{(m)}$.

$$a_{x\bar{t}|}^{(m)} = a_{x\bar{t}|} - \frac{m-1}{2m}\left(1 - \frac{D_{x+t}}{D_x}\right)$$

$$= a_{x\bar{t}|}\left\{1 - \frac{m-1}{2m}\left(\frac{1}{a_{x\bar{t}|}} - P_{x\bar{t}|}^{\,1}\right)\right\}$$

$$\mathrm{a}^{(m)}_{\overline{x\bar{n}|}} = \mathrm{a}_{\overline{x\bar{n}|}}\left\{1 - \frac{m-1}{2m}(\mathrm{P}_{\overline{x\bar{n}|}} + d - \mathrm{P}_{\overline{x\bar{n}|}}^{\ \ 1})\right\}$$

$$= \mathrm{a}_{\overline{x\bar{n}|}}\left\{1 - \frac{m-1}{2m}(\mathrm{P}^1_{\overline{x\bar{n}|}} + d)\right\} \text{ since } \mathrm{P}_{\overline{x\bar{n}|}} = \mathrm{P}^1_{\overline{x\bar{n}|}} + \mathrm{P}_{\overline{x\bar{n}|}}^{\ \ 1}$$

$$\text{Now } \mathrm{P}^{(m)}_{\overline{x\bar{n}|}} = \frac{\mathrm{A}_{\overline{x\bar{n}|}}}{\mathrm{a}^{(m)}_{\overline{x\bar{n}|}}}$$

$$= \frac{\mathrm{A}_{\overline{x\bar{n}|}}}{\mathrm{a}_{\overline{x\bar{n}|}}\left\{1 - \frac{m-1}{2m}(\mathrm{P}^1_{\overline{x\bar{n}|}} + d)\right\}}$$

$$= \frac{\mathrm{P}_{\overline{x\bar{n}|}}}{1 - \frac{m-1}{2m}(\mathrm{P}^1_{\overline{x\bar{n}|}} + d)}$$

From this we obtain

$$\mathrm{P}^{(m)}_{\overline{x\bar{n}|}} = \mathrm{P}_{\overline{x\bar{n}|}} + \frac{m-1}{2m}\mathrm{P}^{(m)}_{\overline{x\bar{n}|}}(\mathrm{P}^1_{\overline{x\bar{n}|}} + d)$$

Here the loss of premium in the event of death of (x) has only to be insured against for the term of the assurance, since if the life survives the term no loss of premium will be incurred. Therefore the annual premium to cover this loss is $\frac{m-1}{2m}\mathrm{P}^{(m)}_{\overline{x\bar{n}|}}\mathrm{P}^1_{\overline{x\bar{n}|}}$.

The addition to cover loss of interest is by the reasoning of the previous problem $\frac{m-1}{2m}\mathrm{P}^{(m)}_{\overline{x\bar{n}|}}d$.

11. To find ${}_t\mathrm{P}^{(m)}_x$.

$${}_t\mathrm{P}^{(m)}_x = \frac{\mathrm{A}_x}{\mathrm{a}^{(m)}_{\overline{x\bar{t}|}}}$$

$$= \frac{\mathrm{A}_x}{\mathrm{a}_{\overline{x\bar{t}|}}\left\{1 - \frac{m-1}{2m}(\mathrm{P}^1_{\overline{x\bar{t}|}} + d)\right\}}$$

$$= \frac{{}_t\mathrm{P}_x}{1 - \frac{m-1}{2m}(\mathrm{P}^1_{\overline{x\bar{t}|}} + d)}$$

$$\text{whence } {}_t\mathrm{P}^{(m)}_x = {}_t\mathrm{P}_x + \frac{m-1}{2m}{}_t\mathrm{P}^{(m)}_x(\mathrm{P}^1_{\overline{x\bar{t}|}} + d)$$

The additions for premium and interest are on the same principles as already explained.

12. The $m$thly premiums we have been discussing are true premiums. Each instalment is $\frac{P^{(m)}}{m}$, and if the life die after only r of these have been paid in any year the office has no claim on the remaining $(m-r)$. But as pointed out in *Text Book*, Article 42, it is sometimes the custom of offices to calculate the premiums in such a way as to make it part of the contract that the unpaid instalments in the year of death shall be deducted from the sum assured; that is, in effect, that they shall all be paid. In such a case the $m$thly premiums are merely instalments of an annual premium which must be paid in full. We must therefore ignore, in the expressions already found, the corrections for loss of premium. We shall then have

$$P_x^{[m]} = P_x + \frac{m-1}{2m} P_x^{[m]} d$$

m being put in square brackets to denote these "instalment" premiums.

Or we may proceed from the beginning as before; on the conditions stated,

$$a_x^{[m]} = a_x \frac{1}{m}\left(1 + v^{\frac{1}{m}} + v^{\frac{2}{m}} + \cdots + v^{\frac{m-1}{m}}\right)$$

$$= \frac{a_x}{m}\left\{1 + (1-d)^{\frac{1}{m}} + (1-d)^{\frac{2}{m}} + \cdots + (1-d)^{\frac{m-1}{m}}\right\}$$

$$= \frac{a_x}{m}\left\{m - d\left(\frac{1}{m} + \frac{2}{m} + \cdots + \frac{m-1}{m}\right)\right\}$$

where we put $(1-d)^{\frac{r}{m}} = 1 - \frac{r}{m}d$ approximately

$$= \frac{a_x}{m}\left(m - \frac{m-1}{2}d\right)$$

$$= a_x\left(1 - \frac{m-1}{2m}d\right)$$

Now $$P_x^{[m]} = \frac{A_x}{\mathrm{a}_x^{[m]}}$$

$$= \frac{A_x}{\mathrm{a}_x\left(1 - \frac{m-1}{2m}d\right)}$$

$$= \frac{P_x}{1 - \frac{m-1}{2m}d}$$

Hence $$P_x^{[m]} = P_x + \frac{m-1}{2m}P_x^{[m]}d$$

Similarly we shall write

$$P_{x\overline{n}|}^{[m]} = P_{x\overline{n}|} + \frac{m-1}{2m}P_{x\overline{n}|}^{[m]}d$$

and $${}_tP_x^{[m]} = {}_tP_x + \frac{m-1}{2m}\,{}_tP_x^{[m]}d$$

EXAMPLES

1. Show that $a_x^{(2)} = a_x + \frac{1}{4} - \frac{i}{8}$ approximately.

By *Text Book* formula (2)

$$a_x^{(2)} = a_x \times \frac{(1+i)^{\frac{1}{2}}+1}{2} + \tfrac{1}{4}(1+i)^{\frac{1}{2}}A_x$$

$$= a_x \times \frac{(1+i)^{\frac{1}{2}}+1}{2} + \tfrac{1}{4}(1+i)^{\frac{1}{2}}\frac{1-ia_x}{1+i}$$

$$= \frac{a_x}{4}\left\{2 + 2(1+i)^{\frac{1}{2}} - i(1+i)^{-\frac{1}{2}}\right\} + \tfrac{1}{4}(1+i)^{-\frac{1}{2}}$$

Neglecting higher powers of i than the first

$$a_x^{(2)} = \frac{a_x}{4}(2+2+i-i) + \tfrac{1}{4}(1-\tfrac{1}{2}i)$$

$$= a_x + \tfrac{1}{4} - \frac{i}{8}$$

2. Given the following formulas :—

$$
\begin{aligned}
&(a)\ A_x = v(1+a_x) - a_x \\
&(b)\ 1 = ia_x + (1+i)A_x \\
&(c)\ A_x = v - iva_x \\
&(d)\ A_x = 1 - d(1+a_x) \\
&(e)\ a_x = a_\infty - (1+a_\infty)A_x ;
\end{aligned}
$$

transform them from yearly to *m*thly intervals and give a verbal interpretation of each result.

$$(a)\quad A_x^{(m)} = mv^{\frac{1}{m}}\left(\frac{1}{m} + a_x^{(m)}\right) - ma_x^{(m)}$$

On the analogy of *Text Book,* Chapter VII., Article 30, we have the explanation as follows :—

The difference between the value of an annuity payable by instalments of $\frac{1}{m}$ at the end of each *m*thly interval of a year upon which (x) enters and that of a similar annuity payable at the end of each such interval which he completes is the value of $\frac{1}{m}$ payable at the end of that *m*thly interval upon which he enters but which he does not complete: which again is the value of an assurance of $\frac{1}{m}$ payable at the end of that *m*thly interval in which (x) dies. But the value of the former annuity is $v^{\frac{1}{m}}\left(\frac{1}{m} + a_x^{(m)}\right)$ and of the latter $a_x^{(m)}$, and of the assurance $\frac{1}{m}\,A_x^{(m)}$, and we therefore have

$$
\begin{aligned}
\frac{1}{m}A_x^{(m)} &= v^{\frac{1}{m}}\left(\frac{1}{m} + a_x^{(m)}\right) - a_x^{(m)} \\
A_x^{(m)} &= mv^{\frac{1}{m}}\left(\frac{1}{m} + a_x^{(m)}\right) - ma_x^{(m)}
\end{aligned}
$$

$$(b)\quad 1 = m\{(1+i)^{\frac{1}{m}} - 1\}a_x^{(m)} + (1+i)^{\frac{1}{m}}A_x^{(m)}$$

See *Text Book,* Chapter VII., Article 41. If 1 be invested now at effective rate i it will yield at the end of each $\frac{1}{m}$ of a year

$\{(1+i)^{\frac{1}{m}}-1\}$ of interest so long as (x) lives, and at the end of that $m$thly interval in which (x) dies we shall have an instalment of interest as above together with the original 1,

$$\{(1+i)^{\frac{1}{m}}-1\}+1 = (1+i)^{\frac{1}{m}}.$$

But the value of an annuity of $\frac{1}{m}$ per interval is $a_x^{(m)}$. Therefore the value of a similar annuity of $\{(1+i)^{\frac{1}{m}}-1\}$ is $m\{(1+i)^{\frac{1}{m}}-1\}a_x^{(m)}$. And the value of the assurance is $(1+i)^{\frac{1}{m}}A_x^{(m)}$. Now these two together make up the whole value of the 1 originally invested and we have accordingly $1 = m\{(1+i)^{\frac{1}{m}}-1\}a_x^{(m)}+(1+i)^{\frac{1}{m}}A_x^{(m)}$.

$$(c)\ A_x^{(m)} = v^{\frac{1}{m}} - m\{(1+i)^{\frac{1}{m}}-1\}v^{\frac{1}{m}}a_x^{(m)}$$

For the explanation of this, see page 209 following.

$$(d)\ A_x^{(m)} = 1 - m\{(1+i)^{\frac{1}{m}}-1\}v^{\frac{1}{m}}\left(\frac{1}{m} + a_x^{(m)}\right)$$

See *Text Book*, Chapter VII., Article 43, for explanation.

$$(e)\ a_x^{(m)} = a_\infty^{(m)} - \left(\frac{1}{m} + a_\infty^{(m)}\right)A_x^{(m)}$$

This is explained in *Text Book*, Articles 13 and 14 of this chapter.

3. Find the present value of an annuity to (x) (payable half-yearly) commencing at £1, the payments to be doubled every 10 years.

This annuity is equivalent to an annuity of 1 commencing now; plus an annuity of 1 commencing on attainment of age $x+10$; plus an annuity of 2 commencing on attainment of age $x+20$; plus an annuity of 4 commencing on attainment of age $x+30$; and so on. Therefore, taking $a_x^{(2)} = a_x + \frac{1}{4}$, we have the whole value

$$= (a_x+\tfrac{1}{4}) + \frac{D_{x+10}}{D_x}(a_{x+10}+\tfrac{1}{4}) + \frac{D_{x+20}}{D_x}2(a_{x+20}+\tfrac{1}{4})$$
$$+ \frac{D_{x+30}}{D_x}4(a_{x+30}+\tfrac{1}{4}) + \cdot\ \cdot\ \cdot$$

4. Use the *Text Book* table at $3\frac{1}{2}$ per cent. to find the half-yearly premium to secure at age thirty-five a double-endowment assurance payable at the end of 10 years or previous death.

Here we may write

$$\frac{P^{(2)}}{2} = \frac{1}{2}\frac{A_{x\overline{n}|}+A_{x\overline{n}|}^{\;\;1}}{a_{x\overline{n}|}^{(2)}}$$

$$= \frac{1}{2}\frac{M_x - M_{x+n} + 2D_{x+n}}{(N_{x-1} - N_{x+n-1}) - \cdot 25(D_x - D_{x+n})}$$

$$= \frac{1}{2}\frac{M_{35} - M_{45} + 2D_{45}}{(N_{34} - N_{44}) - \cdot 25(D_{35} - D_{45})}$$

$$= \frac{1}{2}\frac{9806 - 7769 + 2 \times 16570}{(474131 - 260247) - \cdot 25(25839 - 16570)}$$

$$= \frac{1}{2}\frac{35177}{211567}$$

$$= \cdot 08313.$$

5. Prove that the value of a temporary annuity-due to (x) for n years, payable half-yearly, is approximately $\frac{1}{4}(3\text{a}_{x\overline{n}|} + a_{x\overline{n}|})$.

$$\text{a}_{x\overline{n}|}^{(m)} = \text{a}_{x\overline{n}|} - \frac{m-1}{2m}\left(1 - \frac{D_{x+n}}{D_x}\right)$$

$$= \frac{1}{2m}\left\{2m\text{a}_{x\overline{n}|} - (m-1) + (m-1)\frac{D_{x+n}}{D_x}\right\}$$

$$= \frac{1}{2m}\left\{(m+1)\text{a}_{x\overline{n}|} + (m-1)\left(\text{a}_{x\overline{n}|} - 1 + \frac{D_{x+n}}{D_x}\right)\right\}$$

$$= \frac{1}{2m}\{(m+1)\text{a}_{x\overline{n}|} + (m-1)a_{x\overline{n}|}\}$$

Hence in the particular case

$$\text{a}_{x\overline{n}|}^{(2)} = \tfrac{1}{4}(3\text{a}_{x\overline{n}|} + a_{x\overline{n}|})$$

6. Prove that the value of a temporary annuity to (x) for n years, payable half-yearly, is approximately $\frac{1}{4}(3a_{x\overline{n}|} + \text{a}_{x\overline{n}|})$.

$$
\begin{aligned}
a^{(2)}_{x\overline{n}|} &= a^{(2)}_x - \frac{D_{x+n}}{D_x} a^{(2)}_{x+n} \\
&= (a_x + \tfrac{1}{4}) - \frac{D_{x+n}}{D_x}(a_{x+n} + \tfrac{1}{4}) \text{ approximately} \\
&= a_{x\overline{n}|} + \tfrac{1}{4}\left(1 - \frac{D_{x+n}}{D_x}\right) \\
&= a_{x\overline{n}|} + \tfrac{1}{4}(\mathrm{a}_{x\overline{n}|} - a_{x\overline{n}|}) \\
&= \tfrac{1}{4}(3a_{x\overline{n}|} + \mathrm{a}_{x\overline{n}|})
\end{aligned}
$$

7. Using the ordinary approximate addition to a yearly annuity to make it payable m times a year, prove that for an assurance of 1 on (x) the premium payable m times a year

$$
= \frac{1}{m}\left(\frac{1 - \frac{m-1}{2m} d}{a_x + \frac{m+1}{2m}} - d\right).
$$

The premium payable at the beginning of each mth part of a year for a whole-of-life assurance on (x) is

$$
\begin{aligned}
\frac{P^{(m)}_x}{m} &= \frac{1}{m}\frac{A_x}{\mathrm{a}^{(m)}_x} \\
&= \frac{1}{m}\frac{1 - d\mathrm{a}_x}{\mathrm{a}^{(m)}_x} \\
&= \frac{1}{m}\frac{1 - d\mathrm{a}^{(m)}_x - \frac{m-1}{2m} d}{\mathrm{a}^{(m)}_x}
\end{aligned}
$$

since $\mathrm{a}^{(m)}_x = \mathrm{a}_x - \dfrac{m-1}{2m}$

$$
= \frac{1}{m}\left(\frac{1 - \frac{m-1}{2m} d}{a_x + \frac{m+1}{2m}} - d\right)
$$

since $\mathrm{a}^{(m)}_x = a_x + \dfrac{m+1}{2m}$.

8. Given P_x at 3 per cent. = ·01930, find the corresponding half-yearly and quarterly premiums.

Since $$P_x^{(m)} = \frac{P_x}{1 - \frac{m-1}{2m}(P_x+d)}$$

Therefore $$P_x^{(2)} = \frac{P_x}{1-\frac{1}{4}(P_x+d)}$$
$$= \frac{\cdot01930}{1-\frac{1}{4}(\cdot01930+\cdot02913)}$$
$$= \frac{\cdot01930}{1-\cdot01211}$$
$$= \frac{1930}{98789}$$
$$= \cdot01954$$

and the half-yearly premium $\dfrac{P_x^{(2)}}{2} = \cdot00977$.

Also $$P_x^{(4)} = \frac{P_x}{1-\frac{3}{8}(P_x+d)}$$
$$= \frac{\cdot01930}{1-\cdot01816}$$
$$= \frac{1930}{98184}$$
$$= \cdot01966$$

and the quarterly premium $\dfrac{P_x^{(4)}}{4} = \cdot00492$.

9. An industrial assurance office grants whole-life policies by weekly premiums of 2d. Find what sum assured can be granted at age x, if the agent introducing the business is allowed as commission the whole of the office premiums for the first 13 weeks and 20 per cent. on them thereafter, and the office expenses amount to 20 per cent. on the premiums from the commencement. Assume that premiums, commission, expenses, and claims are on the average paid in the middle of the year, and that the year contains 52 weeks.

The premium of 2d. a week for 52 weeks = 8s. 8d. = $\cdot 4\dot{3}$ of £1. It is assumed to be payable in the middle of the year on the average, and therefore the value of the premiums is

$$= \cdot 4\dot{3} \times \tfrac{1}{2}(\mathrm{a}_x + a_x) \text{ approximately}$$

$$= \cdot 4\dot{3}\,\frac{\mathrm{N}_x + \frac{1}{2}\mathrm{D}_x}{\mathrm{D}_x}$$

Of the premium 20 per cent. for commission and 20 per cent. for expenses has to be allowed for the whole period of the assurance. This being 40 per cent. of the premium, its value is $\cdot 17\dot{3}\,\frac{\mathrm{N}_x + \frac{1}{2}\mathrm{D}_x}{\mathrm{D}_x}$. In addition to this 20 per cent. for commission, a further 80 per cent. (making together the whole premium) has to be allowed for the first thirteen weeks, and this being payable on the average in the middle of the year, according to assumption, its value is $\cdot 08\dot{6}\,\frac{\mathrm{D}_{x+\frac{1}{2}}}{\mathrm{D}_x}$.

Thus the value of the net premium received by the office is

$$\cdot 4\dot{3}\,\frac{\mathrm{N}_x + \frac{1}{2}\mathrm{D}_x}{\mathrm{D}_x} - \cdot 17\dot{3}\,\frac{\mathrm{N}_x + \frac{1}{2}\mathrm{D}_x}{\mathrm{D}_x} - \cdot 08\dot{6}\,\frac{\mathrm{D}_{x+\frac{1}{2}}}{\mathrm{D}_x}$$

If S be the sum assured to be allowed, its value is

$$\mathrm{S}\,\frac{\mathrm{M}_x(1+i)^{\frac{1}{2}}}{\mathrm{D}_x}$$

And, payment and benefit being equal in present value, we get

$$\mathrm{S} = \frac{\cdot 26(\mathrm{N}_x + \frac{1}{2}\mathrm{D}_x) - \cdot 08\dot{6}\,\mathrm{D}_{x+\frac{1}{2}}}{\mathrm{M}_x(1+i)^{\frac{1}{2}}}$$

10. Given a table of temporary life annuities, show how you would find approximately the value of an annuity on (x) payable yearly for $21\frac{1}{2}$ years, the first payment of 1 being due $2\frac{1}{2}$ years hence.

In commutation symbols

$$_{2\frac{1}{2}|}\mathrm{a}_{x:\overline{21\frac{1}{2}|}} = \frac{1}{\mathrm{D}_x}(\mathrm{D}_{x+2\frac{1}{2}} + \mathrm{D}_{x+3\frac{1}{2}} + \mathrm{D}_{x+4\frac{1}{2}} + \cdots + \mathrm{D}_{x+22\frac{1}{2}} + \tfrac{1}{2}\mathrm{D}_{x+23})$$

since an immediate annuity-due for $21\frac{1}{2}$ years means a payment

of 1 at the beginning of each year for 21 years, with a payment of $\frac{1}{2}$ at the end of $20\frac{1}{2}$ years.

Now $\mathrm{D}_{x+2\frac{1}{2}}+\mathrm{D}_{x+3\frac{1}{2}}+\mathrm{D}_{x+4\frac{1}{2}}+\cdots+\mathrm{D}_{x+22\frac{1}{2}}+\frac{1}{2}\mathrm{D}_{x+23}$

$$= \frac{\mathrm{D}_{x+2}+\mathrm{D}_{x+3}}{2}+\frac{\mathrm{D}_{x+3}+\mathrm{D}_{x+4}}{2}+\cdots+\frac{\mathrm{D}_{x+22}+\mathrm{D}_{x+23}}{2}+\frac{\mathrm{D}_{x+23}}{2} \text{ approx.}$$

$$= (\mathrm{D}_{x+1}+\mathrm{D}_{x+2}+\mathrm{D}_{x+3}+\cdots+\mathrm{D}_{x+23})-(\mathrm{D}_{x+1}+\tfrac{1}{2}\mathrm{D}_{x+2})$$

$$= \mathrm{N}_x-\mathrm{N}_{x+23}-\tfrac{1}{2}\{\mathrm{D}_{x+1}+(\mathrm{D}_{x+1}+\mathrm{D}_{x+2})\}$$

Converting the expression into annuities, we have

$$_{2\frac{1}{2}|}\mathrm{a}_{x:\overline{21\frac{1}{2}}|} = a_{x:\overline{23}|}-\tfrac{1}{2}(a_{x:\overline{1}|}+a_{x:\overline{2}|})$$

This shows us to be correct, for the annuity under consideration is one for 23 years but with no payment for a year and a half.

11. If the force of mortality be constant from age x to age $(x+n)$, show that $\bar{a}_{\overline{ym}|}$ may be expressed in the form $\dfrac{c^m-1}{\log c}$ where $y \nless x$ and $y+m \ngtr x+n$.

Let $\mu_y = \mu_{y+1} = \text{etc.} = \mu_{y+m} = -\log r$.

$$\text{Then} \qquad -\frac{d\log l_y}{dy} = -\log r$$

$$\log l_y = y\log r+\log k$$

$$l_y = kr^y$$

$$_tp_y = r^t \ (t \ngtr m)$$

$$\text{and} \quad v^t{}_tp_y = v^tr^t = c^t \text{ where } vr = c.$$

$$\text{But} \qquad \bar{a}_{\overline{ym}|} = \int_0^m v^t{}_tp_y dt$$

$$= \int_0^m c^t\,dt$$

$$= \frac{c^m-1}{\log c}$$

CHAPTER X

Assurances Payable at any other Moment than the End of the Year of Death

1. The general problem discussed is to find $A_x^{(m)}$, the present value of 1 payable at the end of that mth part of a year in which (x) dies.

First, we may make the assumption of the uniform distribution of deaths over each year. Taking any year, say the tth, and dividing it into m equal parts, we say that the chance of (x) dying in any one of these is equal to $\frac{1}{m}$ of the chance of his dying within that year. If then (x) die within the first mth part of the year, 1 is payable at the end thereof, which is equivalent to $(1+i)^{\frac{m-1}{m}}$ payable at the end of the year. Similarly if (x) die in the second mth part, 1 is payable at the end thereof, which is equivalent to $(1+i)^{\frac{m-2}{m}}$ payable at the end of the year, and so on. Therefore, should (x) die in the tth year, the value at the end thereof of 1, payable at the end of that mth part of the year in which he dies, is equal to

$$\frac{1}{m}\left\{(1+i)^{\frac{m-1}{m}}+(1+i)^{\frac{m-2}{m}}+\cdots+(1+i)^{\frac{m-m}{m}}\right\}$$

$$=\frac{i}{m\{(1+i)^{\frac{1}{m}}-1\}}$$

Similarly for every year up to the limit of life; and we may say that an assurance of 1 payable at the end of that mth part of a year in which (x) dies is equivalent to an assurance of $\frac{i}{m\{(1+i)^{\frac{1}{m}}-1\}}$ payable at the end of the year of death, or in symbols

$$A_x^{(m)} = \Sigma v^t{}_{t-1|}q_x \times \frac{i}{m\{(1+i)^{\frac{1}{m}}-1\}}$$

$$= A_x \times \frac{i}{j_{(m)}}$$

where $j_{(m)}$ is the nominal rate of interest convertible m times a year corresponding to effective rate i.

If we make m infinite, we have

$$\bar{A}_x = A_x \times \frac{i}{\delta}$$

2. Again, $A_x^{(m)}$ means that on the average 1 will be payable $\frac{1}{2m}$ of a year after death, and on our original assumption death will occur on the average at the middle of the year. Therefore on the average 1 will be payable $\left(\frac{1}{2}+\frac{1}{2m}\right)$ of a year after the commencement of the year of death.

Therefore

$$A_x^{(m)} = \frac{v^{\left(\frac{1}{2}+\frac{1}{2m}\right)}d_x + v^{\left(1+\frac{1}{2}+\frac{1}{2m}\right)}d_{x+1} + \ldots}{l_x}$$

$$= (1+i)^{\left(\frac{1}{2}-\frac{1}{2m}\right)} \times \frac{vd_x + v^2d_{x+1} + \ldots}{l_x}$$

$$= (1+i)^{\left(\frac{1}{2}-\frac{1}{2m}\right)}A_x$$

And when m is infinite

$$\bar{A}_x = (1+i)^{\frac{1}{2}}A_x$$

3. Now, making no assumption as to the distribution of deaths, we may, on the analogy of

$$A_x = v - iva_x,$$

say that

$$A_x^{(m)} = v^{\frac{1}{m}} - mv^{\frac{1}{m}}\{(1+i)^{\frac{1}{m}} - 1\}a_x^{(m)}$$

We may reason this out as follows. If 1 were payable certainly at the end of the first mth part of a year its value would

be $v^{\frac{1}{m}}$. But as it is not payable till the end of that mth part in which (x) dies we must deduct the value of an annuity of the interest on $v^{\frac{1}{m}}$ for every mth part which (x) completes. Now $\{(1+i)^{\frac{1}{m}}-1\}$ is the interest on 1 for the mth part of a year, and therefore $v^{\frac{1}{m}}\{(1+i)^{\frac{1}{m}}-1\}$ is the interest on $v^{\frac{1}{m}}$ for that period. Again $a_x^{(m)}$ is the value of the annuity which pays $\frac{1}{m}$ at the end of every mth part. Therefore the value of the annuity to provide $v^{\frac{1}{m}}\{(1+i)^{\frac{1}{m}}-1\}$ at the end of every such period is $mv^{\frac{1}{m}}\{(1+i)^{\frac{1}{m}}-1\}a_x^{(m)}$. Hence $A_x^{(m)} = v^{\frac{1}{m}} - mv^{\frac{1}{m}}\{(1+i)^{\frac{1}{m}}-1\}a_x^{(m)}$.

When m is infinite $v^{\frac{1}{m}} = 1$, and $m\{(1+i)^{\frac{1}{m}}-1\} = \delta$; therefore

$$\bar{A}_x = 1 - \delta\bar{a}_x.$$

4. Care must be taken in adapting these formulas to the case of endowment assurances. We have

$$A_{x\overline{n}|} = A^1_{x\overline{n}|} + A_{x\overline{n}|}^{\ \ 1}$$

and on the assumption of a uniform distribution of deaths

$$A^{(m)}_{x\overline{n}|} = A^1_{x\overline{n}|} \times \frac{i}{j_{(m)}} + A_{x\overline{n}|}^{\ \ 1}$$

and $$A^{(m)}_{x\overline{n}|} = A^1_{x\overline{n}|}(1+i)^{\left(\frac{1}{2}-\frac{1}{2m}\right)} + A_{x\overline{n}|}^{\ \ 1}$$

Also $$\bar{A}_{x\overline{n}|} = A^1_{x\overline{n}|} \times \frac{i}{\delta} + A_{x\overline{n}|}^{\ \ 1}$$

and $$\bar{A}_{x\overline{n}|} = A^1_{x\overline{n}|}(1+i)^{\frac{1}{2}} + A_{x\overline{n}|}^{\ \ 1}$$

Again, without any such assumption

$$A^{(m)}_{x\overline{n}|} = v^{\frac{1}{m}} - mv^{\frac{1}{m}}\{(1+i)^{\frac{1}{m}}-1\}a^{(m)}_{x:\overline{n-\frac{1}{m}}|}$$

And $$\bar{A}_{x\overline{n}|} = 1 - \delta\bar{a}_{x\overline{n}|}.$$

5. The proof of *Text Book*, Article 16, may be shown more clearly.

$$\begin{aligned}\bar{A}_x &= \frac{1}{l_x}\int_0^\infty v^t\, l_{x+t}\mu_{x+t}dt \\ &= \frac{1}{l_x}\int_0^\infty v^t\, l_{x+t}\left(-\frac{1}{l_{x+t}}\frac{dl_{x+t}}{dt}\right)dt \\ &= -\frac{1}{l_x}\int_0^\infty v^t\,\frac{dl_{x+t}}{dt}dt\end{aligned}$$

Now by the method of integration by parts

$$\int p\frac{dq}{dx}dx = \int\frac{dpq}{dx}dx - \int q\frac{dp}{dx}dx$$

and $$\int_0^\infty p\frac{dq}{dx}dx = \int_0^\infty\frac{dpq}{dx}dx - \int_0^\infty q\frac{dp}{dx}dx$$

Hence $$\begin{aligned}\int_0^\infty v^t\frac{dl_{x+t}}{dt}dt &= \int_0^\infty\frac{dv^t\, l_{x+t}}{dt}dt - \int_0^\infty l_{x+t}\frac{dv^t}{dt}dt \\ &= -l_x - \int_0^\infty l_{x+t}v^t\log v\,dt \\ &= -l_x + \delta\int_0^\infty v^t\, l_{x+t}dt\end{aligned}$$

And $$\begin{aligned}\bar{A}_x &= -\frac{1}{l_x}\left(-l_x + \delta\int_0^\infty v^t\, l_{x+t}dt\right) \\ &= 1 - \delta\bar{a}_x.\end{aligned}$$

We may also proceed thus :—

Since $$\begin{aligned}\frac{dD_{x+t}}{dt} &= \frac{dv^{x+t}\, l_{x+t}}{dt} \\ &= v^{x+t}\frac{dl_{x+t}}{dt} + l_{x+t}\frac{dv^{x+t}}{dt} \\ &= -v^{x+t}\, l_{x+t}\mu_{x+t} + l_{x+t}v^{x+t}\log v \\ &= -D_{x+t}(\mu_{x+t} + \delta).\end{aligned}$$

We have

$$-\frac{1}{D_x}\int_0^\infty \frac{dD_{x+t}}{dt}dt = \frac{1}{D_x}\int_0^\infty D_{x+t}(\mu_{x+t}+\delta)dt$$

$$= \frac{1}{D_x}\int_0^\infty D_{x+t}\mu_{x+t}dt + \delta\frac{1}{D_x}\int_0^\infty D_{x+t}dt$$

that is $$1 = \bar{A}_x + \delta\bar{a}_x$$

and therefore $$\bar{A}_x = 1 - \delta\bar{a}_x.$$

6. Formula (14) may be deduced as follows, probably more easily than by the method of the *Text Book*.

$$\bar{a}_x = \int_0^\infty v^t{}_tp_x dt$$

Therefore, differentiating both sides with respect to x

$$\frac{d\bar{a}_x}{dx} = \int_0^\infty v^t\left(\frac{d_tp_x}{dx}\right)dt$$

But $$\frac{d_tp_x}{dx} = \frac{d\,\dfrac{l_{x+t}}{l_x}}{dx}$$

$$= \frac{l_x\dfrac{dl_{x+t}}{dx} - l_{x+t}\dfrac{dl_x}{dx}}{(l_x)^2}$$

$$= \frac{-l_x l_{x+t}\mu_{x+t} + l_{x+t}l_x\mu_x}{(l_x)^2}$$

$$= (\mu_x - \mu_{x+t})_tp_x$$

Therefore $$\frac{d\bar{a}_x}{dx} = \int_0^\infty v^t(\mu_x - \mu_{x+t})_tp_x dt$$

$$= \mu_x\int_0^\infty v^t{}_tp_x dt - \int_0^\infty v^t{}_tp_x\mu_{x+t}dt$$

$$= \mu_x\bar{a}_x - \bar{A}_x$$

And $$\bar{A}_x = \mu_x\bar{a}_x - \frac{d\bar{a}_x}{dx}$$

7. Similarly $\bar{a}_{x\overline{n}|} = \int_0^n v^t{}_tp_x dt$

Therefore $\dfrac{d\bar{a}_{x\overline{n}|}}{dx} = \int_0^n v^t\left(\dfrac{d_tp_x}{dx}\right)dt$

$$= \int_0^n v^t(\mu_x - \mu_{x+t})_tp_x dt$$

$$= \mu_x\int_0^n v^t{}_tp_x dt - \int_0^n v^t{}_tp_x\mu_{x+t}dt$$

$$= \mu_x\bar{a}_{x\overline{n}|} - \bar{A}^1_{x\overline{n}|}$$

And $\bar{A}^1_{x\overline{n}|} = \mu_x\bar{a}_{x\overline{n}|} - \dfrac{d\bar{a}_{x\overline{n}|}}{dx}$

8. The following method of deducing formula (20) is perhaps preferable to that indicated in the *Text Book.*

$$\mathring{e}_x = \int_0^\infty {}_tp_x dt$$

Therefore, differentiating both sides,

$$\frac{d\mathring{e}_x}{dx} = \frac{d}{dx}\left(\int_0^\infty {}_tp_x dt\right)$$

$$= \int_0^\infty\left(\frac{d_tp_x}{dx}\right)dt$$

$$= \int_0^\infty(\mu_x - \mu_{x+t})_tp_x dt$$

Hence $= \mu_x\int_0^\infty {}_tp_x dt - \int_0^\infty {}_tp_x\mu_{x+t}dt$

$$= \mu_x\mathring{e}_x - 1$$

And $\mu_x = \dfrac{1}{\mathring{e}_x}\left(1 + \dfrac{d\mathring{e}_x}{dx}\right)$

$$= \frac{1}{\mathring{e}_x}\{1 - \tfrac{1}{2}(\mathring{e}_{x-1} - \mathring{e}_{x+1})\} \text{ approximately.}$$

9. The following three annual premiums should be noted :—

$^{(\infty)}P_x$ is the annual premium payable once a year required to

provide an assurance of 1 payable at the moment of death and accordingly is equal to $\frac{\bar{A}_x}{1+a_x}$.

$\bar{P}_x$ is the annual premium payable by momently instalments to provide an assurance of 1 at the end of the year of death and is equal to $\frac{A_x}{\bar{a}_x}$.

${}^{(\infty)}\bar{P}_x$ is the annual premium payable by momently instalments to provide an assurance of 1 at the moment of death and is equal to $\frac{\bar{A}_x}{\bar{a}_x}$.

EXAMPLES

1. Find the weekly premium required to provide a double-endowment assurance to (x) payable at the end of n years. Provision is to be made for the immediate payment of claims.

The benefit side

$$= \frac{(M_x - M_{x+n})(1+i)^{\frac{1}{2}} + 2D_{x+n}}{D_x}$$

As to the premiums, it is usual in the case of weekly contributions to assume that they are best represented by a continuous annuity, since the interval is so short. The payment side will thus be

$$\begin{aligned} P \times \bar{a}_{\overline{xn}|} &= P(\bar{a}_x - v^n {}_n p_x \bar{a}_{x+n}) \\ &= P\,\frac{(N_x - N_{x+n}) + \frac{1}{2}(D_x - D_{x+n})}{D_x} \end{aligned}$$

where P is the total contribution per annum.

$$\text{Hence} \qquad P = \frac{(M_x - M_{x+n})(1+i)^{\frac{1}{2}} + 2D_{x+n}}{(N_x - N_{x+n}) + \frac{1}{2}(D_x - D_{x+n})}$$

The contribution to be paid at the beginning of each week is therefore $\frac{P}{52}$.

2. Use the *Text Book* table at 3 per cent. to find the value of $\bar{A}_{30:\overline{20}|}$ by three methods.

(*a*) $$\bar{A}_{30:\overline{20}|} = \frac{(M_{30}-M_{50})(1+i)^{\frac{1}{2}}+D_{50}}{D_{30}}$$

$$= \frac{(14462-9409)1{\cdot}015+16605}{36949}\left(\text{taking } (1+i)^{\frac{1}{2}} = 1+\frac{i}{2}\right)$$

$$= {\cdot}58821.$$

(*b*) $$\bar{A}_{30:\overline{20}|} = \frac{(M_{30}-M_{50})\frac{i}{\delta}+D_{50}}{D_{30}}$$

$$= \frac{(14462-9409)\frac{{\cdot}03}{{\cdot}02956}+16605}{36949}$$

$$= {\cdot}58819.$$

(*c*) $$\bar{A}_{30:\overline{20}|} = 1-\delta\bar{a}_{30:\overline{20}|}.$$

We may put $$\bar{a}_{30:\overline{20}|} = \frac{N_{30}-N_{50}+\frac{1}{2}(D_{30}-D_{50})}{D_{30}}$$

$$= \frac{(735104-230450)+\frac{1}{2}(36949-16605)}{36949}$$

$$= 13{\cdot}933.$$

Therefore $$\bar{A}_{30:\overline{20}|} = 1-{\cdot}02956\times 13{\cdot}933$$

$$= {\cdot}58814.$$

3. Find the necessary formulas for the single premiums for the following benefits:—

(*a*) An assurance payable 12 years after the death of (x).

(*b*) An assurance payable at the end of 12 years from entry, provided (x) has died at any time within the 12 years.

(*a*) The required single premium is $v^{12}\bar{A}_x$.

For $\bar{A}_x$, one or other of the following formulas may be taken,

$$\bar{A}_x = A_x(1+i)^{\frac{1}{2}}$$

$$\bar{A}_x = A_x\times\frac{i}{\delta}$$

or $$\bar{A}_x = 1-\delta\bar{a}_x$$

(*b*) For this assurance we have

$$v^{12}\,{}_{|12}q_x = v^{12}(1 - {}_{12}p_x)$$

4. Show that, by a mortality table which follows Makeham's first modification of Gompertz's law, $\bar{A}_x = -\log s\,\bar{a}_x + (\mu_x + \log s)\bar{a}'_x$ where $\bar{a}'_x$ is calculated at a rate of interest j, such that $\dfrac{1}{1+j} = \dfrac{c}{1+i}$.

$$\begin{aligned}
\bar{A}_x &= \frac{1}{l_x}\int_0^\infty v^t\,l_{x+t}\mu_{x+t}\,dt \\
&= \frac{1}{l_x}\int_0^\infty v^t\,l_{x+t}\{-\log s - (\log g\ \log c)c^{x+t}\}\,dt \\
&= -\frac{1}{l_x}\log s\int_0^\infty v^t\,l_{x+t}\,dt - \frac{1}{l_x}(\log g\ \log c)c^x\int_0^\infty (vc)^t\,l_{x+t}\,dt \\
&= -\log s\,\bar{a}_x + (\mu_x + \log s)\bar{a}'_x
\end{aligned}$$

since $\mu_x = -\log s - (\log g\ \log c)c^x$

and therefore $-(\log g\ \log c)c^x = \mu_x + \log s$.

$\bar{a}'_x$ being calculated at the required rate j which is such that $\dfrac{1}{1+j} = \dfrac{c}{1+i}$.

5. Prove that

(*a*) $\dfrac{d}{dx}\bar{a}_x = \bar{a}_x(\mu_x + \delta) - 1$

(*b*) $\dfrac{d}{dx}(l_x\bar{a}_x) = -l_x\bar{A}_x$

$$\text{(a)}\quad \frac{d}{dx}\bar{a}_x = \frac{d}{dx}\left(\frac{\bar{N}_x}{D_x}\right)$$

$$= \frac{D_x\dfrac{d\bar{N}_x}{dx} - \bar{N}_x\cdot\dfrac{dD_x}{dx}}{(D_x)^2}$$

Now
$$\frac{d}{dx}\overline{N}_x = \frac{d}{dx}\int_0^\infty D_{x+t}dt$$
$$= \int_0^\infty\left(\frac{d}{dx}D_{x+t}\right)dt$$
$$= \int_0^\infty\left(\frac{d}{dt}D_{x+t}\right)dt$$
$$= -D_x$$
and
$$\frac{d}{dx}D_x = -D_x(\mu_x+\delta), \text{ (see page 211).}$$
Therefore
$$\frac{d}{dx}\bar{a}_x = \frac{-(D_x)^2+\overline{N}_x D_x(\mu_x+\delta)}{(D_x)^2}$$
$$= \bar{a}_x(\mu_x+\delta)-1.$$
This may be simply found also from the relation
$$\overline{A}_x = \mu_x\bar{a}_x - \frac{d\bar{a}_x}{dx}.$$

(*b*)
$$\frac{d}{dx}(l_x\bar{a}_x) = l_x\frac{d}{dx}\bar{a}_x + \bar{a}_x\frac{d}{dx}l_x$$
$$= l_x\{(\mu_x+\delta)\bar{a}_x - 1\} - l_x\mu_x\bar{a}_x$$
$$= -l_x(1-\delta\bar{a}_x)$$
$$= -l_x\overline{A}_x.$$

6. Prove that

(*a*)
$$\frac{d\overline{M}_x}{dx} = -\mu_x D_x$$

(*b*)
$$\frac{d\overline{A}_x}{dx} = \overline{A}_x(\mu_x+\delta)-\mu_x$$

(*a*) $\overline{A}_x = 1-\delta\bar{a}_x$

Therefore $\overline{M}_x = D_x - \delta\int_0^\infty D_{x+t}dt$

And differentiating both sides
$$\frac{d\overline{M}_x}{dx} = \frac{dD_x}{dx} - \delta\int_0^\infty\left(\frac{dD_{x+t}}{dx}\right)dt$$
$$= \frac{dD_x}{dx} - \delta\int_0^\infty\left(\frac{dD_{x+t}}{dt}\right)dt$$
$$= -D_x(\mu_x+\delta)+\delta D_x$$
$$= -\mu_x D_x$$

$$(b)\ \frac{d\bar{A}_x}{dx} = \frac{d\,\dfrac{\bar{M}_x}{D_x}}{dx}$$

$$= \frac{D_x\dfrac{d\bar{M}_x}{dx} - \bar{M}_x\dfrac{dD_x}{dx}}{(D_x)^2}$$

$$= \frac{-\mu_x(D_x)^2 + \bar{M}_x D_x(\mu_x+\delta)}{(D_x)^2}$$

$$= \bar{A}_x(\mu_x+\delta) - \mu_x$$

7. If the force of mortality be constant ($=c$), prove by direct integration that $\bar{A}_x = \dfrac{c}{c+\delta}$.

Let $c = -\log s = \mu_x = \mu_{x+t}$.

Since $\mu_x = \dfrac{-d\log l_x}{dx}$

then $\dfrac{-d\log l_x}{dx} = -\log s$,

and integrating, $\log l_x = x\log s + \log k$

$$l_x = ks^x$$

$$\text{Now}\quad \bar{A}_x = \frac{1}{l_x}\int_0^\infty v^t\, l_{x+t}\,\mu_{x+t}\,dt$$

$$= \frac{1}{ks^x}\int_0^\infty v^t\, ks^{x+t}(-\log s)\,dt$$

$$= \int_0^\infty v^t s^t(-\log s)\,dt$$

$$= \frac{-\log s}{-\log vs}$$

$$= \frac{-\log s}{-\log s - \log v}$$

$$= \frac{c}{c+\delta}$$

CHAPTER XI

Complete Annuities

1. On the assumption of a uniform distribution of deaths, the value, at the beginning of the year of death, of the correction to the yearly annuity to make it complete is as shown in *Text Book,* Article 5,

$$\frac{1}{r^2}\left(v^{\frac{1}{r}} + 2v^{\frac{2}{r}} + 3v^{\frac{3}{r}} + \ . \ . \ . \ + rv^{\frac{r}{r}}\right)$$

From this is seen clearly the point discussed in *Text Book,* Article 3, viz., that the correction given by formula (1) is too large. The average capital payment is

$$\frac{1}{r^2}(1+2+3+ \ . \ . \ . \ +r)$$

$$= \frac{1}{r^2}\frac{r(r+1)}{2}$$

$$= \frac{r+1}{2r}$$

$$= \tfrac{1}{2} \text{ when } r \text{ is infinite.}$$

But to arrive at the value at the commencement of the year of death, the earlier and smaller payments are multiplied by values ($v^{\frac{1}{r}}$, $v^{\frac{2}{r}}$, $v^{\frac{3}{r}}$, etc.) which are greater than the values ($v^{\frac{r}{r}}$, $v^{\frac{r-1}{r}}$, $v^{\frac{r-2}{r}}$, etc.) by which the later and larger payments are multiplied. Thus the true correction is less than $\frac{1}{2}v^{\frac{1}{2}}$.

$$\text{Now let } s = v^{\frac{1}{r}} + 2v^{\frac{2}{r}} + 3v^{\frac{3}{r}} + \ . \ . \ . \ + rv^{\frac{r}{r}}$$

$$v^{\frac{1}{r}}s = v^{\frac{2}{r}} + 2v^{\frac{3}{r}} + \ . \ . \ . \ + (r-1)v^{\frac{r}{r}} + rv^{\frac{r+1}{r}}$$

$$s(1-v^{\frac{1}{r}}) = v^{\frac{1}{r}}+v^{\frac{2}{r}}+v^{\frac{3}{r}}+\ .\ .\ .\ +v^{\frac{r}{r}}-rv^{\frac{r+1}{r}}$$

$$= v^{\frac{1}{r}}\frac{1-v}{1-v^{\frac{1}{r}}}-rv^{\frac{r+1}{r}}$$

$$s\{(1+i)^{\frac{1}{r}}-1\} = \frac{1-v}{1-v^{\frac{1}{r}}}-rv$$

$$= \frac{iv(1+i)^{\frac{1}{r}}}{\{(1+i)^{\frac{1}{r}}-1\}}-rv$$

$$s = \frac{iv(1+i)^{\frac{1}{r}}}{\{(1+i)^{\frac{1}{r}}-1\}^2} - \frac{rv}{\{(1+i)^{\frac{1}{r}}-1\}}$$

Therefore the value of the correction at the beginning of the year of death is

$$\frac{iv(1+i)^{\frac{1}{r}}}{r^2\{(1+i)^{\frac{1}{r}}-1\}^2} - \frac{v}{r\{(1+i)^{\frac{1}{r}}-1\}}$$

And at the end of the year of death

$$\frac{i(1+i)^{\frac{1}{r}}}{r^2\{(1+i)^{\frac{1}{r}}-1\}^2} - \frac{1}{r\{(1+i)^{\frac{1}{r}}-1\}}$$

Making r infinite we have as the correction

$$\frac{i}{\delta^2} - \frac{1}{\delta}.$$

Multiplying this by A_x and adding the result to a_x we have

$$\mathring{a}_x = a_x+A_x\left(\frac{i}{\delta^2}-\frac{1}{\delta}\right)$$

$$= a_x+A_x\frac{i-\delta}{\delta^2}$$

$$= a_x+\overline{A}_x\frac{i-\delta}{i\delta}$$

since $\overline{A}_x = A_x \times \frac{i}{\delta}$ approximately.

2. In the case of annuities payable m times a year we have only to alter the interval of time from a year to the mth part of a year, and the payment from 1 to $\frac{1}{m}$. If then we break up the mth part of a year into r parts as before, we have as the correction

$$\frac{1}{m}\frac{1}{r^2}\left(v^{\frac{1}{mr}} + 2v^{\frac{2}{mr}} + 3v^{\frac{3}{mr}} + \ . \ . \ . \ + rv^{\frac{r}{mr}}\right)$$

which is precisely analogous to the correction in the case of the yearly annuity. We have merely to alter the rate of interest from i to $\{(1+i)^{\frac{1}{m}} - 1\}$. Accordingly, at the end of the mth part of the year in which death occurs, the value of the correction is

$$\frac{1}{m}\left[\frac{\{(1+i)^{\frac{1}{m}} - 1\}(1+i)^{\frac{1}{mr}}}{r^2\{(1+i)^{\frac{1}{mr}} - 1\}^2} - \frac{1}{r\{(1+i)^{\frac{1}{mr}} - 1\}}\right]$$

But when r is made infinitely great, $(1+i)^{\frac{1}{mr}}$ is unity, and $r\{(1+i)^{\frac{1}{mr}} - 1\}$ is $\log(1+i)^{\frac{1}{m}} = \frac{\delta}{m}$. And writing $\frac{j_{(m)}}{m}$ for $\{(1+i)^{\frac{1}{m}} - 1\}$ we have as the correction

$$\frac{1}{m}\left\{\frac{\frac{j_{(m)}}{m}}{\left(\frac{\delta}{m}\right)^2} - \frac{1}{\frac{\delta}{m}}\right\} = \frac{j_{(m)} - \delta}{\delta^2}$$

Therefore

$$\begin{aligned} \mathring{a}_x^{(m)} &= a_x^{(m)} + A_x^{(m)}\frac{j_{(m)} - \delta}{\delta^2} \\ &= a_x^{(m)} + \overline{A}_x\frac{j_{(m)} - \delta}{j_{(m)}\delta} \end{aligned}$$

since $\overline{A}_x = A_x^{(m)} \times \frac{j_{(m)}}{\delta}$ approximately.

3. Making no assumption as to distribution of deaths, and taking r very great and approaching infinity, we have as the value of the correction in respect of the $(n+1)$th year for a yearly annuity

$$\frac{1}{D_x}\left\{\frac{1}{r}(\bar{M}_{x+n} - \bar{M}_{x+n+\frac{1}{r}}) + \frac{2}{r}(\bar{M}_{x+n+\frac{1}{r}} - \bar{M}_{x+n+\frac{2}{r}}) + \cdots \right.$$
$$\left. + \frac{r}{r}(\bar{M}_{x+n+\frac{r-1}{r}} - \bar{M}_{x+n+1})\right\}$$
$$= \frac{1}{D_x}\left\{\frac{1}{r}(\bar{M}_{x+n} + \bar{M}_{x+n+\frac{1}{r}} + \cdots + \bar{M}_{x+n+\frac{r-1}{r}}) - \bar{M}_{x+n+1}\right\}$$

Now

$$\bar{M}_{x+n} = \bar{M}_{x+n}$$

$$\bar{M}_{x+n+\frac{1}{r}} = \bar{M}_{x+n} + \frac{1}{r}\Delta\bar{M}_{x+n} + \frac{\frac{1}{r}\left(\frac{1}{r}-1\right)}{2}\Delta^2\bar{M}_{x+n} + \cdots$$

etc. etc. etc.

$$\bar{M}_{x+n+\frac{r-1}{r}} = \bar{M}_{x+n} + \frac{r-1}{r}\Delta\bar{M}_{x+n} + \frac{\frac{r-1}{r}\left(\frac{r-1}{r}-1\right)}{2}\Delta^2\bar{M}_{x+n} + \cdots$$

Therefore $\Sigma = r\bar{M}_{x+n} + \frac{r-1}{2}\Delta\bar{M}_{x+n} - \frac{r^2-1}{12r}\Delta^2\bar{M}_{x+n} + \cdots$

where $\Delta\bar{M}_{x+n} = \bar{M}_{x+n+1} - \bar{M}_{x+n}$, etc.

Hence if we stop at second differences the correction is

$$\frac{1}{D_x}\left\{\frac{1}{r}\left(r\bar{M}_{x+n} + \frac{r-1}{2}\Delta\bar{M}_{x+n} - \frac{r^2-1}{12r}\Delta^2\bar{M}_{x+n}\right) - \bar{M}_{x+n+1}\right\}$$
$$= \frac{1}{D_x}\left(\bar{M}_{x+n} + \frac{r-1}{2r}\Delta\bar{M}_{x+n} - \frac{r^2-1}{12r^2}\Delta^2\bar{M}_{x+n} - \bar{M}_{x+n+1}\right)$$

(making r infinitely great)

$$= \frac{1}{D_x}\left(\bar{M}_{x+n} - \bar{M}_{x+n+1} + \frac{1}{2}\Delta\bar{M}_{x+n} - \frac{1}{12}\Delta^2\bar{M}_{x+n}\right)$$

Giving to n every value from 0 to $\omega - x$ and adding the correction to a_x we have

$$\mathring{a}_x = a_x + \frac{1}{2}\frac{\bar{M}_x}{D_x} + \frac{1}{12}\frac{\Delta\bar{M}_x}{D_x}$$
$$= a_x + \frac{1}{2}\bar{A}_x - \frac{1}{12}\bar{A}^{1}_{x:\overline{1}|}$$

4. Again, in the case of annuities payable m times a year, we have only to alter the interval of time from a year to the mth part of a year and the payment from 1 to $\frac{1}{m}$. If then we break up the $\left(n+\frac{k+1}{m}\right)$th $m$thly interval into r parts, r being very great and approaching infinity, the value of the correction is

$$\frac{1}{m}\frac{1}{D_x}\left\{\frac{1}{r}(\bar{M}_{x+n+\frac{k}{m}} - \bar{M}_{x+n+\frac{k}{m}+\frac{1}{mr}}) + \frac{2}{r}(\bar{M}_{x+n+\frac{k}{m}+\frac{1}{mr}} - \bar{M}_{x+n+\frac{k}{m}+\frac{2}{mr}}) + \cdots \right.$$
$$\left. + \frac{r}{r}(\bar{M}_{x+n+\frac{k}{m}+\frac{r-1}{mr}} - \bar{M}_{x+n+\frac{k+1}{m}})\right\}$$

$$= \frac{1}{mD_x}\left(\bar{M}_{x+n+\frac{k}{m}} + \frac{r-1}{2r}\Delta\bar{M}_{x+n+\frac{k}{m}} - \frac{r^2-1}{12r^2}\Delta^2\bar{M}_{x+n+\frac{k}{m}} - \bar{M}_{x+n+\frac{k+1}{m}}\right)$$

by a process similar to the above, where

$$\Delta\bar{M}_{x+n+\frac{k}{m}} = \bar{M}_{x+n+\frac{k+1}{m}} - \bar{M}_{x+n+\frac{k}{m}}, \text{ etc.}$$

Now making r infinite we have as the value

$$\frac{1}{mD_x}(\bar{M}_{x+n+\frac{k}{m}} - \bar{M}_{x+n+\frac{k+1}{m}} + \tfrac{1}{2}\Delta\bar{M}_{x+n+\frac{k}{m}} - \tfrac{1}{12}\Delta^2\bar{M}_{x+n+\frac{k}{m}})$$

and giving to $\left(n+\frac{k}{m}\right)$ every value from 0 upwards we have as the value of the total correction

$$\frac{1}{mD_x}\{\tfrac{1}{2}\bar{M}_x + \tfrac{1}{12}(\bar{M}_{x+\frac{1}{m}} - \bar{M}_x)\}$$

But $\bar{M}_{x+\frac{1}{m}} = \bar{M}_x + \frac{1}{m}(\bar{M}_{x+1} - \bar{M}_x)$ approximately;

therefore $$\frac{\bar{M}_{x+\frac{1}{m}} - \bar{M}_x}{D_x} = \frac{\frac{1}{m}(\bar{M}_{x+1} - \bar{M}_x)}{D_x}$$
$$= -\frac{1}{m}\bar{A}^1_{x:\overline{1|}}$$

Hence $$\bar{a}^{(m)}_x = a^{(m)}_x + \frac{1}{2m}\bar{A}_x - \frac{1}{12m^2}\bar{A}^1_{x:\overline{1|}}$$

Or by a better approximation

$$\overline{M}_{x+\frac{1}{m}} = \overline{M}_x + \frac{1}{m}\frac{d\overline{M}_x}{dx}$$

$$\text{and} \quad \frac{d\overline{M}_x}{dx} = -\mu_x D_x$$

(See Chapter X. of these notes, Example 6 (a).)

$$\text{Therefore} \quad \frac{\overline{M}_{x+\frac{1}{m}} - \overline{M}_x}{D_x} = -\frac{\mu_x}{m}$$

$$\text{and} \quad \mathring{a}_x^{(m)} = a_x^{(m)} + \frac{1}{2m}\overline{A}_x - \frac{\mu_x}{12m^2}$$

5. We may obtain expressions for complete temporary annuities as follows:—

$$\begin{aligned}
\mathring{a}_{x\overline{n}|} &= \mathring{a}_x - v^n{}_np_x\mathring{a}_{x+n} \\
&= \{a_x + \tfrac{1}{2}(1+i)^{\frac{1}{2}}A_x\} - v^n{}_np_x\{a_{x+n} + \tfrac{1}{2}(1+i)^{\frac{1}{2}}A_{x+n}\} \\
&= a_{x\overline{n}|} + \tfrac{1}{2}(1+i)^{\frac{1}{2}}A^1_{x\overline{n}|} \\
\mathring{a}^{(m)}_{x\overline{n}|} &= \mathring{a}^{(m)}_x - v^n{}_np_x\mathring{a}^{(m)}_{x+n} \\
&= \left\{a_x + \frac{m-1}{2m} + \frac{1}{2m}(1+i)^{\frac{1}{2}}A_x\right\} - v^n{}_np_x\left\{a_{x+n} + \frac{m-1}{2m} + \frac{1}{2m}(1+i)^{\frac{1}{2}}A_{x+n}\right\} \\
&= a_{x\overline{n}|} + \frac{m-1}{2m}\left(1 - \frac{D_{x+n}}{D_x}\right) + \frac{1}{2m}(1+i)^{\frac{1}{2}}A^1_{x\overline{n}|}
\end{aligned}$$

These approximations are close enough for practical purposes, as the greater accuracy of the more refined formulas is to a certain extent inoperative on account of the subtractive portion in the above formulas.

EXAMPLES

1. A life annuity of P, payable by half-yearly instalments, was purchased at age x. After n years (a half-year's payment having just been made) it is desired to make the annuity payable by quarterly instalments in future and with a proportion to date of death. To what sum must the annuity be reduced if the alteration be given effect to?

Remembering that $\mathring{a}_x^{(m)} = a_x^{(m)} + \frac{1}{2m}A_x(1+i)^{\frac{1}{2}}$, and that $a_x^{(m)} = a_x + \frac{m-1}{2m}$, we have the value of the annuity as at present constituted

$$= P(\tfrac{1}{4} + a_{x+n}).$$

If K be the annual payment in future under the conditions required, the value of the future payments

$$= K\{\tfrac{3}{8} + a_{x+n} + \tfrac{1}{8}A_{x+n}(1+i)^{\frac{1}{2}}\}.$$

Hence, equating

$$K\{\tfrac{3}{8} + a_{x+n} + \tfrac{1}{8}A_{x+n}(1+i)^{\frac{1}{2}}\} = P(\tfrac{1}{4} + a_{x+n})$$

and
$$K = \frac{P(\frac{1}{4} + a_{x+n})}{\frac{3}{8} + a_{x+n} + \frac{1}{8}A_{x+n}(1+i)^{\frac{1}{2}}}.$$

2. Given that £100 will purchase an annuity of £5·026 payable yearly, find the corresponding annuity also payable yearly but with a proportion to the date of death, using 3 per cent. interest.

Here
$$a_x = \frac{100}{5\cdot026}$$
$$= 19\cdot897.$$

And
$$\mathring{a}_x = a_x + \tfrac{1}{2}A_x(1+i)^{\frac{1}{2}}$$
$$= a_x + \tfrac{1}{2}\{1 - d(1+a_x)\}\left(1 + \frac{i}{2}\right)$$
$$\text{taking } (1+i)^{\frac{1}{2}} = 1 + \frac{i}{2}$$
$$= 19\cdot897 + \tfrac{1}{2}(1 - \cdot02913 \times 20\cdot897)(1\cdot015)$$
$$= 19\cdot897 + \cdot199$$
$$= 20\cdot096.$$

Therefore £100 purchases a complete annuity of

$$\frac{100}{20\cdot096} = 4\cdot976.$$

3. A man aged 70 has £250, which he decides to invest in an annuity. What annuity, payable quarterly in advance and with a proportion to date of death, can be allowed to him, given $a_{70} = 7\cdot299$ and d at 3 per cent. $= \cdot02913$, and assuming a loading of 10 per cent. to be added to the pure value?

It is necessary to find the value of $\mathring{a}^{(4)}_{70}$ in the first place.

$$\text{Now} \qquad \mathring{a}^{(m)}_x = a^{(m)}_x + \frac{1}{2m}A_x(1+i)^{\frac{1}{2}} \text{ by formula (2)}$$

$$= a_x + \frac{m+1}{2m} + \frac{1}{2m}\{1 - d(1+a_x)\}(1+i)^{\frac{1}{2}}$$

$$\text{Therefore } \mathring{a}^{(4)}_{70} = a_{70} + \tfrac{5}{8} + \tfrac{1}{8}\{1 - d(1+a_{70})\}(1+i)^{\frac{1}{2}}$$

$$= 7{\cdot}299 + {\cdot}625 + {\cdot}125(1 - {\cdot}02913 \times 8{\cdot}299)1{\cdot}015$$

$$\left((1+i)^{\frac{1}{2}} \text{ being taken as } 1 + \frac{i}{2} = 1{\cdot}015\right)$$

$$= 7{\cdot}924 + {\cdot}096$$

$$= 8{\cdot}020.$$

Adding the loading of 10 per cent. we get 8·822 as the price of an annuity of 1. Therefore £250 will purchase an annuity of $\frac{250}{8{\cdot}822} = 28{\cdot}338$.

The annuity, payable quarterly in advance and with a proportion to date of death, which can be purchased is therefore £28, 6s. 9d.

4. Show that $\bar{A}_x = 1 - i\bar{a}_x$ approximately.

$$\text{We have} \qquad A_x = \frac{1 - ia_x}{1+i}$$

$$\text{whence} \qquad (1+i)A_x = 1 - ia_x$$

$$\text{But} \qquad (1+i) = (1+i)^{\frac{1}{2}}(1+i)^{\frac{1}{2}}$$

$$= (1+i)^{\frac{1}{2}}\left(1 + \frac{i}{2}\right) \text{ approximately.}$$

$$\text{Therefore} \qquad (1+i)^{\frac{1}{2}}A_x\left(1 + \frac{i}{2}\right) = 1 - ia_x$$

$$\text{and} \qquad (1+i)^{\frac{1}{2}}A_x = 1 - ia_x - \frac{i}{2}(1+i)^{\frac{1}{2}}A_x$$

$$= 1 - i\{a_x + \tfrac{1}{2}(1+i)^{\frac{1}{2}}A_x\}$$

$$\text{that is} \qquad \bar{A}_x = 1 - i\bar{a}_x \text{ approximately.}$$

5. Show that $\ddot{a}_x^{(m)} = v^{\frac{1}{2m}}\bar{a}_x$ approximately.

$$
\begin{aligned}
\ddot{a}_x^{(m)} &= a_x + \frac{m-1}{2m} + \frac{1}{2m}\overline{A}_x \\
&= (a_x + \tfrac{1}{2}) - \frac{1}{2m}(1 - \overline{A}_x) \\
&= \bar{a}_x - \frac{1}{2m}\delta\bar{a}_x \\
&= \bar{a}_x\left(1 - \frac{d}{2m}\right) \text{ since } \delta = d \text{ approximately,} \\
&= v^{\frac{1}{2m}}\bar{a}_x \text{ approximately,}
\end{aligned}
$$

since

$$
\begin{aligned}
v^{\frac{1}{2m}} &= (1-d)^{\frac{1}{2m}} \\
&= 1 - \frac{d}{2m} \text{ approximately.}
\end{aligned}
$$

CHAPTER XII

Joint-Life Annuities

1. In a table in which Gompertz's Law holds we have fundamentally

$$\mu_x = Bc^x$$

and $\quad l_x = kg^{c^x}$

whence $\quad {}_tp_x = g^{c^x(c^t-1)}$

and $\quad a_{xy} = \Sigma v^t {}_tp_{xy}$

$$= \Sigma v^t g^{(c^x+c^y)(c^t-1)}$$

(putting $c^x+c^y = c^w$)

$$= \Sigma v^t g^{c^w(c^t-1)}$$

$$= \Sigma v^t {}_tp_w$$

$$= a_w$$

Similarly $a_{xyz} = a_w$ where $c^x+c^y+c^z = c^w$, and generally $a_{xyz\ldots(m)} = a_w$ where $c^x+c^y+c^z+\cdots$ to m terms $= c^w$.

Now, if $c^x+c^y+c^z+\cdots$ to m terms $= c^w$,

then $\quad Bc^x+Bc^y+Bc^z+\cdots$ to m terms $= Bc^w$,

that is, $\quad \mu_x+\mu_y+\mu_z+\cdots$ to m terms $= \mu_w$.

From this we see that in the case of a table following Gompertz's Law we can find the values of joint-life annuities, provided we have a table of the force of mortality and of annuities for single lives. If, for example, we wish to find the value of $a_{30:40:50}$, we first obtain w such that $\mu_w=\mu_{30}+\mu_{40}+\mu_{50}$ and then a_w at the given rate of interest is the value required.

2. Again, in a table in which Makeham's first modification of Gompertz's law holds we have

$$\mu_x = A + Bc^x$$
$$l_x = ks^x g^{c^x}$$
and $$_tp_x = s^t g^{c^x(c^t-1)}$$
Hence $$a_{xy} = \Sigma v^t {}_tp_{xy}$$
$$= \Sigma v^t s^{2t} g^{(c^x+c^y)(c^t-1)}$$
(putting $c^x + c^y = 2c^w$)
$$= \Sigma v^t s^{2t} g^{2c^w(c^t-1)}$$
$$= \Sigma v^t {}_tp_{ww}$$
$$= a_{ww}$$

Similarly $a_{xyz} = a_{www}$ where $c^x + c^y + c^z = 3c^w$, and generally $a_{xyz \ldots (m)} = a_{www \ldots (m)}$ where $c^x + c^y + c^z + \cdots$ to m terms $= mc^w$.

Now, if $c^x + c^y + c^z + \cdots$ to m terms $= mc^w$,

then $Bc^x + Bc^y + Bc^z + \cdots$ to m terms $= mBc^w$,

and $(A + Bc^x) + (A + Bc^y) + (A + Bc^z) + \cdots$ to m terms $= m(A + Bc^w)$,

that is, $\mu_x + \mu_y + \mu_z + \cdots$ to m terms $= m\mu_w$.

We thus see that we can find the values of joint-life annuities in such a table, provided we are supplied with a table of the force of mortality and with tables of annuities on joint lives of equal ages. For example, to find $a_{30:40:50}$, we first find w such that $3\mu_w = \mu_{30} + \mu_{40} + \mu_{50}$ and then a_{www} at the given rate of interest is the value required.

3. If in the relation $2c^w = c^x + c^y$ we assume that $x < y$ we have

$$2c^w = c^x(1 + c^{y-x})$$
$$c^{w-x} = \frac{1 + c^{y-x}}{2}$$
and $$w - x = \frac{\log(1 + c^{y-x}) - \log 2}{\log c}$$

from which we see that the value of $w-x$ is the same for all values of x and y where the difference between x and y is constant. In other words, the addition to be made to the younger age to find the equivalent equal age is constant where $(y-x)$ is constant. We might therefore form such a table as the following:—

$y-x$	$w-x = \dfrac{\log(1+c^{y-x})-\log 2}{\log c}$
1	
2	
3	
4	
5	
etc.	

Entering this table with the difference between the two ages, we find, in the second column opposite, the addition to be made to the younger age to obtain the equivalent equal ages.

If there be three lives we have

$$3c^u = c^x + c^y + c^z.$$

where $x < y < z$.

From the above table find w such that $2c^w = c^x + c^y$. Then

$$\begin{aligned} 3c^u &= 2c^w + c^z \\ &= c^w(2 + c^{z-w}) \\ c^{u-w} &= \frac{2 + c^{z-w}}{3} \\ u - w &= \frac{\log(2 + c^{z-w}) - \log 3}{\log c} \end{aligned}$$

We might then form a second table of which the first column should be integral values of $z-w$ and the second should be the corresponding values of the above expression which is equal to $u-w$. Then for annuity values, etc., involving three lives we should find from the first table the value of $w-x$ corresponding to $y-x$ and hence find w; and thereafter find from the second table the value of $u-w$ corresponding to $z-w$ and hence find u. The value of the required function is then that of a similar function on three lives all aged u.

4. Still considering a table subject to Makeham's first modification of Gompertz's law we have as before

$$a_{xy} = \Sigma v^t {}_tp_{xy}$$

$$= \Sigma v^t s^{2t} g^{(c^x+c^y)(c^t-1)}$$

(putting $c^x + c^y = c^w$)

$$= \Sigma v^t s^t s^t g^{c^w(c^t-1)}$$

$$= \Sigma v'^t {}_tp_w$$

$$= a'_w$$

where a'_w is calculated at such a rate j, that $\dfrac{1}{1+j} = \dfrac{s}{1+i}$.

Similarly $a_{xyz} = \Sigma v^t s^{2t} s^t g^{c^w(c^t-1)}$

$$= \Sigma v'^t {}_tp_w$$

$$= a'_w$$

where $c^w = c^x + c^y + c^z$ and the annuity is calculated at a rate j such that $\dfrac{1}{1+j} = \dfrac{s^2}{1+i}$.

Generally $a_{xyz \dots (m)} = a'_w$
where $c^w = c^x + c^y + c^z + \cdots$ to m terms, and the annuity is calculated at a rate j such that $\dfrac{1}{1+j} = \dfrac{s^{m-1}}{1+i}$.

The problem may be stated still more generally.

$$a_{xyz \dots (m)} = \Sigma v^t s^{mt} g^{(c^x+c^y+c^z+\cdots \text{ to } m \text{ terms})(c^t-1)}$$

putting $c^x + c^y + c^z + \cdots$ to m terms $= rc^w$

$$= \Sigma v^t s^{(m-r)t} s^{rt} g^{rc^w(c^t-1)}$$

$$= \Sigma v'^t {}_tp_{www \dots (r)}$$

$$= a'_{www \dots (r)}$$

calculated at a rate of interest j such that $\dfrac{1}{1+j} = \dfrac{s^{m-r}}{1+i}$.

In practice it would be most convenient to make $r = 1$, as, under this second method, tables of annuity values have to be calculated at special rates depending on the number of joint lives, and these special tables will of course be most easily prepared for single-life annuities. A table of c^x for all values of x must also be prepared, as a table of μ_x in this connection is inconvenient.

5. A constant increase in the force of mortality under Makeham's law has the effect of an increase in the rate of interest. For if in the expression

$$\mu_x = -\log s - (\log g \ \log c)c^x$$

we add a constant $-\log r$, where r is a positive fraction and consequently $-\log r$ is also positive, we have

$$\mu'_x = -(\log s + \log r) - (\log g \ \log c)c^x$$

whence simply

$$\begin{aligned} l'_x &= ks^x r^x g^{c^x} \\ &= r^x l_x \\ {}_tp'_x &= r^t \, {}_tp_x \end{aligned}$$

Also the value of an annuity on (x) in the new table is

$$\begin{aligned} a'_{x(i)} &= \Sigma v^t \, {}_tp'_x \\ &= \Sigma v^t r^t \, {}_tp_x \\ &= a_{x(j)} \end{aligned}$$

where $a_{x(j)}$ is calculated at a special rate j which is such that $\frac{1}{1+j} = \frac{r}{1+i}$. From this we see that $\frac{1}{1+j} < \frac{1}{1+i}$ since r is a positive fraction; and consequently $j > i$. It may be mentioned that an increase of ·01 in the force of mortality is very nearly equivalent to a rise of 1 per cent. in the rate of interest.

Further, in any table, as indicated on page 211,

$$\mu_x + \delta = -\frac{1}{D_x}\frac{dD_x}{dx} = -\frac{d\log D_x}{dx}$$

Now in a table where μ_x is increased to $\mu_x - \log r$

$$-\frac{d\log D'_x}{dx} = (\mu_x - \log r) + \delta$$

Also in a table where δ is increased to $\delta - \log r$

$$-\frac{d\log D''_x}{dx} = \mu_x + (\delta - \log r)$$

Therefore $$-\frac{d\log D'_x}{dx} = -\frac{d\log D''_x}{dx}$$

$$D'_x = D''_x \text{ for all values of } x$$

and $$a'_x = a''_x.$$

Thus a constant addition to the force of mortality is equivalent to the same constant addition to the force of interest. From this fact the practical assumption is made that a constant addition, say ·01, to the rate of mortality is equivalent to the same constant addition, 1 per cent., to the rate of interest.

Though $a'_{(i)}$ by the extra mortality table is equal to $a_{(j)}$ by the normal, it does not follow that the corresponding single and annual premiums are also equal.

$$\begin{aligned}
\text{For} \qquad A'_{(i)} &= 1 - d_{(i)}(1 + a'_{(i)}) \\
&= 1 - d_{(i)}(1 + a_{(j)}) \text{ since } a'_{(i)} = a_{(j)} \\
\text{whereas} \qquad A_{(j)} &= 1 - d_{(j)}(1 + a_{(j)}) \\
\text{Again} \qquad P'_{(i)} &= \frac{1}{1 + a'_{(i)}} - d_{(i)} \\
&= \frac{1}{1 + a_{(j)}} - d_{(i)} \\
\text{But} \qquad P_{(j)} &= \frac{1}{1 + a_{(j)}} - d_{(j)}
\end{aligned}$$

6. An increase in the constant B has the same effect as increasing the age. For if $\mu_x = A + Bc^x$, let

$$\mu'_x = A + B'c^x$$

where $B' > B$.

Find h such that $B' = Bc^h$,

$$\begin{aligned}
\text{Then} \qquad \mu'_x &= A + Bc^h c^x \\
&= A + Bc^{x+h} \\
&= \mu_{x+h}
\end{aligned}$$

7. In a table which follows Makeham's first modification, when it is required to find the value of the annuity $a_{\overline{xyz\ldots(m)}}^{\;r}$, it is not correct to put it equal to $a_{\overline{www\ldots(m)}}^{\;r}$ where $m\mu_w = \mu_x + \mu_y + \mu_z + \cdots$ to m terms.

We must proceed more slowly.

$$\begin{aligned}
a_{\overline{xyz\ldots(m)}}^{\;r} &= \frac{Z^r}{(1+Z)^r} \\
&= Z^r - rZ^{r+1} + \frac{r(r+1)}{2}Z^{r+2} - \ldots
\end{aligned}$$

Or, to take a particular case,

$$a_{\overline{wxyz}} = (a_w + a_x + a_y + a_z) - (a_{wx} + a_{wy} + a_{wz} + a_{xy} + a_{xz} + a_{yz})$$
$$+ (a_{wxy} + a_{wxz} + a_{wyz} + a_{xyz}) - a_{wxyz}$$

Here we must determine separately by the above rule the values of all the joint-life annuities involved and hence find the value of the annuity in question, for it is only to joint-life annuities that the rule applies.

8. As already pointed out in the notes on Chapter VI., under Makeham's second modification of Gompertz's law, the expression for the force of mortality takes the form

$$\mu_x = \mathrm{A} + \mathrm{H}x + \mathrm{B}c^x$$

whence simply $\quad l_x = ks^x w^{x^2} g^{c^x}$

and $\quad {}_tp_x = s^t w^{2xt+t^2} g^{c^x(c^t-1)}$

also $\quad {}_tp_{x+d} = s^t w^{2t(x+d)+t^2} g^{c^{x+d}(c^t-1)}$

Therefore $\quad {}_tp_{x:x+d} = s^{2t} w^{2t(2x+d)+2t^2} g^{(c^x+c^{x+d})(c^t-1)}$

Now putting $\quad c^x + c^{x+d} = 2c^z$

we have $\quad c^{z-x} = \dfrac{1+c^d}{2}$

and $\quad z - x = \dfrac{\log(1+c^d) - \log 2}{\log c}$

$$= d'$$

where d' depends on d, the difference between the ages of (x) and $(x+d)$.

Also $\quad x = z - d'$

Hence $\quad {}_tp_{x:x+d} = s^{2t} w^{2t(2z-2d'+d)+2t^2} g^{2c^z(c^t-1)}$

$$= s^{2t} w^{2t \cdot 2z + 2t^2} g^{2c^z(c^t-1)} w^{2t(d-2d')}$$

$$= {}_tp_{zz} w^{2t(d-2d')}$$

But $\quad a_{x:x+d} = \Sigma v^t {}_tp_{x:x+d}$

$$= \Sigma v^t w^{2t(d-2d')} {}_tp_{zz}$$

$$= a'_{zz}$$

where a'_{zz} is calculated at a rate j such that

$$\frac{1}{1+j} = \frac{w^{2(d-2d')}}{1+i}$$

Again $\quad {}_tp_{x:x+d:x+e} = s^{3t} w^{2t(3x+d+e)+3t^2} g^{(c^x+c^{x+d}+c^{x+e})(c^t-1)}$

And putting $\quad 3c^z = c^x+c^{x+d}+c^{x+e}$

we have $\quad c^{z-x} = \dfrac{1+c^d+c^e}{3}$

and $\quad z-x = \dfrac{\log(1+c^d+c^e)-\log 3}{\log c}$

$$= d'$$

where d' depends on d and e the differences between the ages.

Hence $\quad {}_tp_{x:x+d:x+e} = s^{3t} w^{2t(3z-3d'+d+e)+3t^2} g^{3c^z(c^t-1)}$

$$= {}_tp_{zzz} w^{2t(d+e-3d')}$$

and $\quad a_{x:x+d:x+e} = a'_{zzz}$

where a'_{zzz} is calculated at rate j which is such that

$$\frac{1}{1+j} = \frac{w^{2(d+e-3d')}}{1+i}$$

Generally $\quad a_{x:x+d:x+e:\dots(m)} = a'_{zzz\dots(m)}$

where $a'_{zzz\dots(m)}$ is calculated at rate j which is such that

$$\frac{1}{1+j} = \frac{w^{2(d+e+\dots\text{ to }(m-1)\text{ terms}-md')}}{1+i}$$

and $\quad d' = \dfrac{\log(1+c^d+c^e+\dots\text{ to } m \text{ terms})-\log m}{\log c}$

It will be seen that, under this second modification, tables of joint-life annuities would be required calculated at special rates of interest depending on the number of lives and on the differences between the youngest and all other lives. A table of c^z is also necessary.

EXAMPLES

1. Obtain an approximation to $a_{50:70:75}$ by the Carlisle Table at $3\frac{1}{2}$ per cent.

(*a*) Simpson's rule.

$$\begin{aligned} a_{70:75} &= 3\cdot731 \\ &= a_{82\cdot411} \\ a_{50:82} &= 3\cdot562 \\ a_{50:83} &= 3\cdot369 \\ a_{50:82\cdot411} \text{ or } a_{50:70:75} &= 3\cdot483 \end{aligned}$$

(*b*) Price's correction.

$$\begin{aligned} a_{50:70:75} &= a_{50:82} - \cdot05 \\ &= 3\cdot562 - \cdot05 \\ &= 3\cdot512 \end{aligned}$$

(*c*) Milne's correction.

$$\begin{aligned} a_{50:70:75} &= a_{50:82\cdot5} \\ &= \tfrac{1}{2}(3\cdot562 + 3\cdot369) \\ &= 3\cdot466 \end{aligned}$$

2. Use Simpson's rule to find by the H^{M} Table at $3\frac{1}{2}$ per cent. the value of $a_{\overline{85:67:78}}$.

$$a_{\overline{85:67:78}} = a_{85} + a_{67} + a_{78} - a_{85:67} - a_{85:78} - a_{67:78} + a_{85:67:78}$$

To find $a_{85:67:78}$ we have

$$\begin{aligned} a_{67:78} &= 4\cdot020 \\ &= a_{78\cdot578} \\ \text{and } a_{85:78} &= 3\cdot997 \\ a_{85:79} &= 3\cdot765 \\ \text{therefore } \quad a_{85:78\cdot578} &= 3\cdot864 \end{aligned}$$

Hence

$$\begin{aligned} a_{\overline{85:67:78}} &= 17\cdot325 + 7\cdot471 + 5\cdot512 - 6\cdot993 - 5\cdot238 - 4\cdot020 + 3\cdot864 \\ &= 17\cdot921 \end{aligned}$$

3. Find from the *Text Book* table the values of $a_{\overline{48:48:59}}^{[2]}$ at 3 per cent., and of $A_{\overline{81:40:44}}$ at $3\frac{1}{2}$ per cent.

$$a_{\overline{48:48:59}}^{[2]} = a_{48:48} + a_{48:59} + a_{48:59} - 3a_{48:48:59}$$

$$= a_{45\cdot78:45\cdot78} + a_{58\cdot69:58\cdot69} + a_{54\cdot88:54\cdot88} - 3a_{52\cdot11:52\cdot11:52\cdot11}$$

$$= 11\cdot915 + 9\cdot284 + 8\cdot901 - 23\cdot910$$

$$= 6\cdot190$$

$$A_{\overline{81:40:44}} = 1 - d(1 + a_{\overline{81:40:44}})$$

$$a_{\overline{81:40:44}} = a_{81} + a_{40} + a_{44} - a_{81:40} - a_{81:44} - a_{40:44} + a_{81:40:44}$$

$$= a_{81} + a_{40} + a_{44} - a_{86\cdot40:86\cdot40} - a_{89\cdot88:89\cdot88} - a_{42\cdot18:42\cdot18} + a_{89\cdot56:89\cdot56:89\cdot56}$$

$$= 18\cdot235 + 16\cdot103 + 14\cdot997 - 13\cdot936 - 13\cdot185 - 12\cdot409 + 11\cdot196$$

$$= 21\cdot001$$

$$A_{\overline{81:40:44}} = 1 - \cdot03382 \times 22\cdot001$$

$$= \cdot25593$$

4. State how you would apply Simpson's rule to obtain the value of $a_{80:47:60:\overline{15}|}$.

From a consideration of the statements in *Text Book*, Article 6, it will appear that it would be incorrect to adopt such a method as the following:—Find w such that $a_w = a_{47:60}$; then $a_{80:w:\overline{15}|}$ is the value required.

It is necessary to split up the temporary annuity into its two parts, and we then have

$$a_{80:47:60:\overline{15}|} = a_{80:47:60} - \frac{D_{45:62:75}}{D_{80:47:60}} a_{45:62:75}$$

Then find w such that $a_w = a_{47:60}$ and w' such that $a_{w'} = a_{62:75}$, and we shall have

$$a_{80:47:60:\overline{15}|} = a_{80:w} - \frac{D_{45}\, l_{62}\, l_{75}}{D_{80}\, l_{47}\, l_{60}} a_{45:w'}$$

5. Show that doubling the force of mortality is equivalent to taking two lives of equal age in the expectation of life, the life annuity, and the assurance.

Let $\mu'_x = 2\mu_x$

$$\text{Then}\quad \frac{d\log l'_x}{dx} = 2\frac{d\log l_x}{dx}$$

$$\log l'_x = 2\log l_x + \log k$$

$$l'_x = k(l_x)^2$$

$$\text{Now}\quad e'_x = \Sigma_t p'_x$$

$$= \Sigma \frac{l'_{x+t}}{l'_x}$$

$$= \Sigma \frac{l_{x+t}}{l_x}\frac{l_{x+t}}{l_x}$$

$$= e_{xx}$$

$$\text{Similarly}\quad a'_x = a_{xx}$$

$$\text{Also}\quad A'_x = v(1+a'_x) - a'_x$$

$$= v(1+a_{xx}) - a_{xx}$$

$$= A_{xx}$$

6. Show that if $l_x = ks^x g^{c^x}$, then $c^y \frac{d}{dx} a_{xy} = c^x \frac{d}{dy} a_{xy}$

$$a_{xy} = \Sigma v^t s^{2t} g^{c^x(c^t-1)} g^{c^y(c^t-1)}$$

$$\text{and}\quad \frac{d}{dx} a_{xy} = \Sigma v^t s^{2t} g^{c^y(c^t-1)} \frac{d}{dx} g^{c^x(c^t-1)}$$

Putting $c^x(c^t-1) = u$, we have

$$\frac{d}{dx} g^{c^x(c^t-1)} = \frac{d}{dx} g^u$$

$$= \frac{d}{du} g^u \frac{du}{dx}$$

$$= g^u \log g \frac{d}{dx} c^x(c^t-1)$$

$$= g^{c^x(c^t-1)} \log g (c^t-1) c^x \log c$$

$$\text{Therefore}\quad c^y \frac{d}{dx} a_{xy} = (\log g \log c) c^{x+y} \Sigma v^t s^{2t} (c^t-1) g^{c^y(c^t-1)} g^{c^x(c^t-1)}$$

Similarly $c^x \dfrac{d}{dy} a_{xy} = (\log g \ \log c) c^{y+x} \Sigma v^t s^{2t} (c^t - 1) g^{c^x(c^t-1)} g^{c^y(c^t-1)}$

Hence $c^y \dfrac{d}{dx} a_{xy} = c^x \dfrac{d}{dy} a_{xy}.$

7. Given in the case of a table following Makeham's rule that

$$\log l_{x+t} = \log k + (x+t) \log s + c^{x+t} \log g$$

show what modifications must be introduced in order that the equation may apply to select tables without loss of the property of uniform seniority.

To obtain an expression for $\log l_{[x]+t}$, instead of $\log k$ and $\log g$ we must write $\log k_t$ and $\log g_t$, since these will vary progressively for each year of duration. Log s and c will remain constant for all durations. We then have

$$\begin{aligned} \log l_{[x]+t} &= \log k_t + (x+t) \log s + c^{x+t} \log g_t \\ \text{and} \quad \log l_{[x]} &= \log k_0 + x \log s + c^x \log g_0 \\ \text{whence } \log {}_tp_{[x]} &= \log l_{[x]+t} - \log l_{[x]} \\ &= (\log k_t - \log k_0 + t \log s) + c^x (c^t \log g_t - \log g_0) \end{aligned}$$

Similarly $\log {}_tp_{[y]} = (\log k_t - \log k_0 + t \log s) + c^y (c^t \log g_t - \log g_0)$

Therefore

$$\log {}_tp_{[x][y]} = 2(\log k_t - \log k_0 + t \log s) + (c^x + c^y)(c^t \log g_t - \log g_0)$$

(putting $2c^w = c^x + c^y$)

$$\begin{aligned} &= 2(\log k_t - \log k_0 + t \log s) + 2c^w (c^t \log g_t - \log g_0) \\ &= \log {}_tp_{[w][w]} \end{aligned}$$

Therefore further $a_{[x][y]} = \Sigma v^t {}_tp_{[x][y]}$

$$\begin{aligned} &= \Sigma v^t {}_tp_{[w][w]} \\ &= a_{[w][w]} \end{aligned}$$

where $\quad c^x + c^y = 2c^w$

Similarly also $a_{[x][y][z]\cdots(m)} = a_{[w][w][w]\cdots(m)}$

where $c^x + c^y + c^z + \cdots$ to m terms $= mc^w$.

It may similarly be proved that

$$a_{[x]+r:[y]+r:[z]+r\cdots(m)} = a_{[w]+r:[w]+r:[w]+r\cdots(m)}$$

where the same relation holds.

CHAPTER XIII

Contingent, or Survivorship, Assurances

1. We may easily transform *Text Book* formula (1) so that it shall be suitable for application to select tables.

$$A^1_{xy} = \frac{1}{2}\left(A_{xy} + \frac{a_{x-1:y}}{p_{x-1}} - \frac{a_{x:y-1}}{p_{y-1}}\right)$$

But $a_{x-1:y} = vp_{x-1:y}(1 + a_{x:y+1})$ and $a_{x:y-1} = vp_{x:y-1}(1 + a_{x+1:y})$

Therefore $\quad A^1_{xy} = \frac{1}{2}\{A_{xy} + vp_y(1 + a_{x:y+1}) - vp_x(1 + a_{x+1:y})\}$

Hence $A^1_{[x][y]} = \frac{1}{2}\{A_{[x][y]} + vp_{[y]}(1 + a_{[x]:[y]+1}) - vp_{[x]}(1 + a_{[x]+1:[y]})\}$

Again, since $A_{xy} = 1 - d(1 + a_{xy})$, we may put the formula in a form for finding the value of A^1_{xy}, given tables of joint-life annuities.

$$A^1_{xy} = \tfrac{1}{2}\{1 - d(1 + a_{xy}) + vp_y(1 + a_{x:y+1}) - vp_x(1 + a_{x+1:y})\}$$

2. In calculating the necessary joint-life annuities or the suitable commutation columns as described in *Text Book*, Article 17, it is not at all essential that both (x) and (y) be taken from the same mortality experience. The two lives are of quite different classes, and the risk will be most accurately calculated by taking their mortality from different tables. The standard basis at the present time is to take the $O^{[af]}$ table for (y) who is in the position of an annuitant or life-tenant, and the $O^{[NM]}$ table for (x) who comes into property on the death of (y), and desires to insure against his dying before that event.* Tables of A^1_{xy} and P^1_{xy} on this basis have been calculated by

* See, however, the word of caution on pp. vi. and vii. of *British Offices' Life Tables*, 1893; "Select Tables, Whole-Life Assurances—Males."

Messrs Baker and Raisin. Further, it is wrong to calculate the values from an aggregate table, for (x) is usually the younger life, and in such a case his mortality would be underestimated, while that of (y), the older, would be overestimated, both errors operating against the office.

3. *Text Book* formula (7) may be shown simply thus:—

$$\begin{aligned}
A^2_{xy} &= \Sigma v^n {}_{n-1|}q^2_{xy} \\
&= \Sigma v^n ({}_{n-1|}q_x - {}_{n-1|}q^1_{xy}) \\
&= \Sigma v^n {}_{n-1|}q_x - \Sigma v^n {}_{n-1|}q^1_{xy} \\
&= A_x - A^1_{xy}
\end{aligned}$$

$$\begin{aligned}
\text{Again} \quad A^{\;2}_{xy} &= \Sigma v^n {}_{n-1|}q^{\;2}_{xy} \\
&= \Sigma v^n ({}_{n-1|}q_y - {}_{n-1|}q^{\;1}_{xy}) \\
&= \Sigma v^n ({}_{n-1|}q_y - {}_{n-1|}q_{xy} + {}_{n-1|}q^1_{xy}) \\
&= A_y - A_{xy} + A^1_{xy}
\end{aligned}$$

All the *Text Book* formulas (9) may be similarly deduced, but by summing from 1 to n only.

$$\begin{aligned}
\text{Thus} \quad {}_{|n}A^{\;1}_{xy} &= \Sigma_1^n v^n {}_{n-1|}q^{\;1}_{xy} \\
&= \Sigma_1^n v^n ({}_{n-1|}q_{xy} - {}_{n-1|}q^1_{xy}) \\
&= {}_{|n}A_{xy} - {}_{|n}A^1_{xy}
\end{aligned}$$

4. An alternative formula for ${}_{|n}A^1_{xy}$ may be found as follows:—

$$\begin{aligned}
{}_{|n}A^1_{xy} &= A^1_{xy} - v^n {}_n p_{xy} A_{\overset{1}{x+n}:\overline{y+n}} \\
&= \frac{1}{2}\left(A_{xy} - \frac{a_{x:y-1}}{p_{y-1}} + \frac{a_{x-1:y}}{p_{x-1}}\right) \\
&\qquad - \frac{1}{2} v^n {}_n p_{xy}\left(A_{x+n:y+n} - \frac{a_{x+n:y+n-1}}{p_{y+n-1}} + \frac{a_{x+n-1:y+n}}{p_{x+n-1}}\right) \\
&= \frac{1}{2}\left({}_{|n}A_{xy} - \frac{{}_{|n}a_{x:y-1}}{p_{y-1}} + \frac{{}_{|n}a_{x-1:y}}{p_{x-1}}\right)
\end{aligned}$$

5. The *Text Book* formulas (8) may best be obtained by deducting the corresponding values in (9) from the whole benefits. Thus:—

$$\begin{aligned} {}_{n|}A^{2}_{xy} &= A^{2}_{xy} - {}_{|n}A^{2}_{xy} \\ &= (A_x - A^{1}_{xy}) - ({}_{|n}A_x - {}_{|n}A^{1}_{xy}) \\ &= {}_{n|}A_x - {}_{n|}A^{1}_{xy} \end{aligned}$$

6. The application of Davies's and De Morgan's types of joint-life commutation symbols respectively to the case of A^{1}_{xy} may be shown more clearly than in *Text Book*, Articles 14 and 15, in the following manner:—

We have $A^{1}_{xy} = \dfrac{1}{l_x l_y} \Sigma v^t d_{x+t-1}\, l_{y+t-\frac{1}{2}}$

Now under Davies's form, where $x > y$, we may write this

$$\frac{\Sigma v^{x+t} d_{x+t-1}\, l_{y+t-\frac{1}{2}}}{D_{xy}} = \frac{\Sigma v^{x+t}(l_{x+t-1} - l_{x+t})(l_{y+t-1} + l_{y+t})}{2\,D_{xy}}$$

$$= \frac{\Sigma v^{x+t}(l_{x+t-1}\, l_{y+t-1} + l_{x+t-1}\, l_{y+t} - l_{x+t}\, l_{y+t-1} - l_{x+t}\, l_{y+t})}{2\,D_{xy}}$$

$$= \frac{v(N_{x-1:y-1} + N_{x-1:y}) - (N_{x:y-1} + N_{xy})}{2\,D_{xy}}$$

Also where $y > x$ we have

$$\frac{\Sigma v^{y+t}(l_{x+t-1}\, l_{y+t-1} + l_{x+t-1}\, l_{y+t} - l_{x+t}\, l_{y+t-1} - l_{x+t}\, l_{y+t})}{2\,D_{xy}}$$

$$= \frac{v(N_{x-1:y-1} - N_{x:y-1}) + (N_{x-1:y} - N_{xy})}{2\,D_{xy}}$$

Under De Morgan's form we shall write

$$\frac{\Sigma v^{\frac{x+y}{2}+t}(l_{x+t-1}\, l_{y+t-1} + l_{x+t-1}\, l_{y+t} - l_{x+t}\, l_{y+t-1} - l_{x+t}\, l_{y+t})}{2\,D_{xy}}$$

$$= \frac{(vN_{x-1:y-1} - N_{xy}) + v^{\frac{1}{2}}(N_{x-1:y} - N_{x:y-1})}{2\,D_{xy}}$$

7. To find $A^1_{x:y(\overline{t|})}$.

$$A^1_{x:y(\overline{t|})} = \frac{vd_x + v^2 d_{x+1} + \cdot \cdot \cdot + v^t d_{x+t-1}}{l_x}$$

$$+ \frac{v^{t+1} d_{x+t}\, l_{y+\frac{1}{2}} + v^{t+2} d_{x+t+1}\, l_{y+1\frac{1}{2}} + \cdot \cdot \cdot}{l_x\, l_y}$$

$$= A^1_{x\overline{t|}} + \frac{v^t\, l_{x+t}}{l_x} \times \frac{vd_{x+t}\, l_{y+\frac{1}{2}} + v^2 d_{x+t+1}\, l_{y+1\frac{1}{2}} + \cdot \cdot \cdot}{l_{x+t}\, l_y}$$

$$= A^1_{x\overline{t|}} + \frac{D_{x+t}}{D_x} A_{\overset{1}{x+t}:y}$$

In practice the following approximation is sometimes used for finding $A^1_{x:y(\overline{t|})}$: Find z such that $e_z = e_y + t$, and then write $A^1_{x:y(\overline{t|})} = A^1_{xz}$.

8. To get the annual premium in this approximation we should write $P^1_{x:y(\overline{t|})} = \dfrac{A^1_{xz}}{1 + a_{xz}}$.

To find the true value of $P^1_{x:y(\overline{t|})}$ we have the divisor of $A^1_{x:y(\overline{t|})}$ either (*a*) $1 + a_{x:y(\overline{t|})}$ or (*b*) $1 + a_{xy}$ as the *Text Book* points out.

(*a*) If we take $P^1_{x:y(\overline{t|})} = \dfrac{A^1_{x:y(\overline{t|})}}{1 + a_{x:y(\overline{t|})}}$, an option is given which is not allowed for in the calculation. For if (y) die early, (x) may be able to secure an equivalent benefit at his then age for a less premium than that payable under this contract. Thus if (y) die, say, at the end of the nth year where n is small, it is possible that $P_{\overset{1}{x+n}:\overline{t|}} < P^1_{x:y(\overline{t|})}$. A minor option may also be exercised in respect of the fact that the last year's premium covers on the average only six months' insurance.

(*b*) On the other hand, if we make $P^1_{x:y(\overline{t|})} = \dfrac{A^1_{x:y(\overline{t|})}}{1 + a_{xy}}$, a risk of another kind is being run. For if (y) be on his deathbed and die after but one payment of premium, the office is granting a t-years term assurance for a quite inadequate single premium.

The former method is probably the better.

9. To find the value of an assurance payable only in the event of (x) dying within t years after the death of (y).

The value of the assurance

$$= \frac{vd_x(l_y - l_{y+\frac{1}{2}}) + v^2 d_{x+1}(l_y - l_{y+1\frac{1}{2}}) + \cdots + v^t d_{x+t-1}(l_y - l_{y+t-\frac{1}{2}})}{l_x l_y}$$

$$+ \frac{v^{t+1} d_{x+t}(l_{y+\frac{1}{2}} - l_{y+t+\frac{1}{2}}) + v^{t+2} d_{x+t+1}(l_{y+1\frac{1}{2}} - l_{y+t+1\frac{1}{2}}) + \cdots}{l_x l_y}$$

$$= \frac{vd_x + v^2 d_{x+1} + \cdots + v^t d_{x+t-1}}{l_x}$$

$$- \frac{vd_x l_{y+\frac{1}{2}} + v^2 d_{x+1} l_{y+1\frac{1}{2}} + \cdots + v^t d_{x+t-1} l_{y+t-\frac{1}{2}} + v^{t+1} d_{x+t} l_{y+t+\frac{1}{2}} + \cdots}{l_x l_y}$$

$$+ \frac{v^t l_{x+t}}{l_x} \times \frac{vd_{x+t} l_{y+\frac{1}{2}} + v^2 d_{x+t+1} l_{y+1\frac{1}{2}} + \cdots}{l_{x+t} l_y}$$

$$= A^1_{x\overline{t}|} - A^1_{xy} + \frac{D_{x+t}}{D_x} A^{\;1}_{x+t:y}$$

$$= A^1_{x:\overline{y(\overline{t}|)}} - A^1_{xy}.$$

This result is obviously correct: for the desired assurance is equivalent to an assurance payable should (x) die before (y) or within t years after the death of (y), less the assurance payable should (x) die before (y).

10. $$A^1_{xyz} = \Sigma v^n \frac{d_{x+n-1}}{l_x} \frac{l_{y+n-\frac{1}{2}}}{l_y} \frac{l_{z+n-\frac{1}{2}}}{l_z}$$

$$= \Sigma v^n \frac{l_{x+n-1} - l_{x+n}}{l_x} \frac{l_{y+n-1} + l_{y+n}}{2l_y} \frac{l_{z+n-1} + l_{z+n}}{2l_z}$$

$$= \Sigma \tfrac{1}{4} v^n \left(\frac{l_{x+n-1} l_{y+n-1} l_{z+n-1}}{l_x l_y l_z} - \frac{l_{x+n} l_{y+n-1} l_{z+n-1}}{l_x l_y l_z} \right.$$

$$+ \frac{l_{x+n-1} l_{y+n-1} l_{z+n}}{l_x l_y l_z} - \frac{l_{x+n} l_{y+n-1} l_{z+n}}{l_x l_y l_z}$$

$$+ \frac{l_{x+n-1} l_{y+n} l_{z+n-1}}{l_x l_y l_z} - \frac{l_{x+n} l_{y+n} l_{z+n-1}}{l_x l_y l_z}$$

$$\left. + \frac{l_{x+n-1} l_{y+n} l_{z+n}}{l_x l_y l_z} - \frac{l_{x+n} l_{y+n} l_{z+n}}{l_x l_y l_z} \right)$$

$$= \frac{1}{4} \left(A_{xyz} - \frac{a_{x:y-1:z-1}}{p_{y-1:z-1}} + \frac{a_{x-1:y-1:z}}{p_{x-1:y-1}} - \frac{a_{x:y-1:z}}{p_{y-1}} + \frac{a_{x-1:y:z-1}}{p_{x-1:z-1}} \right.$$

$$\left. - \frac{a_{x:y:z-1}}{p_{z-1}} + \frac{a_{x-1:y:z}}{p_{x-1}} \right)$$

since $$\Sigma v^n\left(\frac{l_{x+n-1}\,l_{y+n-1}\,l_{z+n-1}}{l_x l_y l_z} - \frac{l_{x+n}\,l_{y+n}\,l_{z+n}}{l_x l_y l_z}\right) = A_{xyz}$$

Formula (21) of the *Text Book* exceeds the above result by

$$\frac{1}{12}\left(A_{xyz} - \frac{a_{x:y-1:z-1}}{p_{y-1:z-1}} - \frac{a_{x-1:y-1:z}}{p_{x-1:y-1}} - \frac{a_{x-1:y:z-1}}{p_{x-1:z-1}} + \frac{a_{x:y-1:z}}{p_{y-1}}\right.$$
$$\left. + \frac{a_{x:y:z-1}}{p_{z-1}} + \frac{a_{x-1:y:z}}{p_{x-1}}\right)$$

which, however, is a small quantity.

The above expression may also be written in the form

$$A^1_{xyz} = \tfrac{1}{4}\{A_{xyz} - vp_x(1 + a_{x+1:y:z}) + vp_y(1 + a_{x:y+1:z}) + vp_z(1 + a_{x:y:z+1})$$
$$- vp_{xy}(1 + a_{x+1:y+1:z}) - vp_{xz}(1 + a_{x+1:y:z+1}) + vp_{yz}(1 + a_{x:y+1:z+1})\}$$

since $a_{x:y-1:z-1} = vp_{x:y-1:z-1}(1 + a_{x+1:y:z})$, etc.

In this form the expression may be applied to select tables.

11. To prove *Text Book* formulas (22) to (27) inclusive we have

$$A^2_{\substack{xyz\\1}} = \Sigma v^n \frac{d_{x+n-1}}{l_x}\,\frac{l_y - l_{y+n-\frac{1}{2}}}{l_y}\,\frac{l_{z+n-\frac{1}{2}}}{l_z}$$

$$= \Sigma v^n \frac{d_{x+n-1}\,l_{z+n-\frac{1}{2}}}{l_x l_z} - \Sigma v^n \frac{d_{x+n-1}\,l_{y+n-\frac{1}{2}}\,l_{z+n-\frac{1}{2}}}{l_x l_y l_z}$$

$$= A^1_{xz} - A^1_{xyz}$$

$$A^2_{xyz} = \Sigma v^n \frac{d_{x+n-1}}{l_x}\left(\frac{l_y - l_{y+n-\frac{1}{2}}}{l_y}\,\frac{l_{z+n-\frac{1}{2}}}{l_z} + \frac{l_z - l_{z+n-\frac{1}{2}}}{l_z}\,\frac{l_{y+n-\frac{1}{2}}}{l_y}\right)$$

$$= \Sigma v^n \frac{d_{x+n-1}}{l_x}\left(\frac{l_{y+n-\frac{1}{2}}}{l_y} + \frac{l_{z+n-\frac{1}{2}}}{l_z} - 2\,\frac{l_{y+n-\frac{1}{2}}\,l_{z+n-\frac{1}{2}}}{l_y l_z}\right)$$

$$= \Sigma v^n \frac{d_{x+n-1}}{l_x}\,\frac{l_{y+n-\frac{1}{2}}}{l_y} + \Sigma v^n \frac{d_{x+n-1}}{l_x}\,\frac{l_{z+n-\frac{1}{2}}}{l_z}$$
$$- 2\Sigma v^n \frac{d_{x+n-1}}{l_x}\,\frac{l_{y+n-\frac{1}{2}}}{l_y}\,\frac{l_{z+n-\frac{1}{2}}}{l_z}$$

$$= A^1_{xy} + A^1_{xz} - 2A^1_{xyz}$$

$$A^{3}_{xyz} = \Sigma v^n \frac{d_{x+n-1}}{l_x}\,\frac{l_y - l_{y+n-\frac{1}{2}}}{l_y}\,\frac{l_z - l_{z+n-\frac{1}{2}}}{l_z}$$

$$= \Sigma v^n \frac{d_{x+n-1}}{l_x} - \Sigma v^n \frac{d_{x+n-1}}{l_x}\,\frac{l_{y+n-\frac{1}{2}}}{l_y} - \Sigma v^n \frac{d_{x+n-1}}{l_x}\,\frac{l_{z+n-\frac{1}{2}}}{l_z}$$

$$+ \Sigma v^n \frac{d_{x+n-1}}{l_x}\,\frac{l_{y+n-\frac{1}{2}}}{l_y}\,\frac{l_{z+n-\frac{1}{2}}}{l_z}$$

$$= A_x - A^1_{xy} - A^1_{xz} + A^1_{xyz}$$

$$A^1_{x:\overline{yz}} = \Sigma v^n \frac{d_{x+n-1}}{l_x}\left(\frac{l_{y+n-\frac{1}{2}}}{l_y}\,\frac{l_z - l_{z+n-\frac{1}{2}}}{l_z} + \frac{l_{z+n-\frac{1}{2}}}{l_z}\,\frac{l_y - l_{y+n-\frac{1}{2}}}{l_y}\right.$$

$$\left.+ \frac{l_{y+n-\frac{1}{2}}}{l_y}\,\frac{l_{z+n-\frac{1}{2}}}{l_z}\right)$$

$$= \Sigma v^n \frac{d_{x+n-1}}{l_x}\left(\frac{l_{y+n-\frac{1}{2}}}{l_y} + \frac{l_{z+n-\frac{1}{2}}}{l_z} - \frac{l_{y+n-\frac{1}{2}}\,l_{z+n-\frac{1}{2}}}{l_y\,l_z}\right)$$

$$= A^1_{xy} + A^1_{xz} - A^1_{xyz}$$

(It should be noted that therefore $A^1_{x:\overline{yz}} = A_x - A^3_{xyz}$)

$$A^{\;\;1}_{\overline{xy}|:z} = \Sigma v^n \frac{l_{z+n-\frac{1}{2}}}{l_z}\left(\frac{d_{x+n-1}}{l_x}\,\frac{l_{y+n-\frac{1}{2}}}{l_y} + \frac{d_{y+n-1}}{l_y}\,\frac{l_{x+n-\frac{1}{2}}}{l_x}\right)$$

$$= A^1_{xyz} + A^{\;1}_{xyz}$$

$$A_{\overline{xy}:\overset{z}{2}} = \Sigma v^n \frac{l_{z+n-\frac{1}{2}}}{l_z}\left(\frac{d_{x+n-1}}{l_x}\,\frac{l_y - l_{y+n-\frac{1}{2}}}{l_y} + \frac{d_{y+n-1}}{l_y}\,\frac{l_x - l_{x+n-\frac{1}{2}}}{l_x}\right)$$

$$= \Sigma v^n \frac{l_{z+n-\frac{1}{2}}}{l_z}\left(\frac{d_{x+n-1}}{l_x} + \frac{d_{y+n-1}}{l_y} - \frac{d_{x+n-1}\,l_{y+n-\frac{1}{2}}}{l_x\,l_y} - \frac{d_{y+n-1}\,l_{x+n-\frac{1}{2}}}{l_y\,l_x}\right)$$

$$= A^1_{xz} + A^1_{yz} - A^1_{xyz} - A^{\;1}_{xyz}$$

12. To find P^2_{xy}

$$P^2_{xy} = \frac{A_x - A^1_{xy}}{1 + a_x}$$

In granting such an assurance by annual premiums, (y) must be medically examined as well as (x). For were (y) in bad health and about to die, the office would be granting a whole-life assurance to (x) for P^2_{xy}, which is less than P_x.

13. To find the annual premiums corresponding to the assurances in *Text Book* formulas (22) to (27) inclusive.

In obtaining the statuses for annual premiums, care must be exercised that premiums are not taken into account beyond the period when they certainly cease to be payable by reason either of the benefit being paid or of the chance of its payment having passed.

$$P^{2}_{\underset{1}{xyz}} = \frac{A^{2}_{\underset{1}{xyz}}}{1+a_{xz}}$$

Here both (x) and (y) must be medically examined.

$$P^{2}_{xyz} = \frac{A^{2}_{xyz}}{1+a_{x:\overline{yz}}}$$

$$P^{3}_{xyz} = \frac{A^{3}_{xyz}}{1+a_{x}}$$

In these two cases all three lives must be examined.

$$P^{1}_{x:\overline{yz}} = \frac{A^{1}_{x:\overline{yz}}}{1+a_{x:\overline{yz}}}$$

The same difficulty arises as in the case of $P^{1}_{x:\overline{y(t|)}}$, since, *e.g.*, if (z) die early, say during the tth year, then possibly $P^{1}_{x+t:\overline{y+t}} < P^{1}_{x:\overline{yz}}$, and an option may be exercised against the office. The alternative is to make $P^{1}_{x:\overline{yz}} = \dfrac{A^{1}_{x:\overline{yz}}}{1+a_{xyz}}$ with a corresponding risk of granting the benefit at an insufficient single premium, if (z), say, be on deathbed. (x) alone is medically examined.

$$P_{\overset{1}{\overline{|xy|}}:z} = \frac{A_{\overset{1}{\overline{|xy|}}:z}}{1+a_{xyz}}$$

$$P_{\underset{3}{\overline{xy}:z}} = \frac{A_{\underset{3}{\overline{xy}:z}}}{1+a_{\overline{xy}:z}}$$

In the last two cases (x) and (y) must be examined.

14. Another and probably better method than that suggested in the *Text Book* of applying Simpson's rule to the calculation of $A^{1}_{x:\overline{yz}}$ and $A_{\underset{3}{\overline{xy}:z}}$ may be pointed out.

$$A^{1}_{x:\overline{yz}} = A^{1}_{xy} + A^{1}_{xz} - A^{1}_{xyz}$$

Then finding w such that $a_w = a_{yz}$ we may write

$$A^1_{x:\overline{yz}} = A^1_{xy} + A^1_{xz} - A^1_{xw}$$

Also $A_{\overline{xy}:\overset{}{\underset{2}{z}}} = A^1_{xz} + A^1_{yz} - A^1_{xyz} - A_{\overset{1}{x}yz}$

Find w such that $a_w = a_{yz}$, and w' such that $a_{w'} = a_{xz}$; and we have $A_{\overline{xy}:\underset{2}{z}} = A^1_{xz} + A^1_{yz} - A^1_{xw} - A^1_{yw'}$.

15. To find the single and annual premiums for an assurance payable on the death of the survivor of two children, ten and fifteen years old respectively, provided both die before attaining age twenty-one during the lifetime of their mother, aged fifty.

The single premium is

$$_{|11}A^1_{10:50} + {}_{|6}A^1_{15:50} - \left({}_{|6}A_{\overline{\overline{10:15}}:50}^{\;1} + \frac{D_{16:21:56}}{D_{10:15:50}}\,{}_{|5}A^1_{16:56}\right)$$

To obtain the annual premium divide this expression by

$$_{|6}a_{10:50} + {}_{|6}a_{15:50} - {}_{|6}a_{10:15:50} + v^6{}_6p_{10:50}(1 - {}_6p_{15})\,{}_{|5}a_{16:56}$$

16. To find $A^1_{\overline{ab}:\overline{xy}}$

$$\begin{aligned}
A^1_{\overline{ab}:\overline{xy}} &= A^1_{a:\overline{xy}} + A^1_{b:\overline{xy}} - A^1_{\overline{\overline{ab}}:\overline{xy}} \\
&= (A^1_{ax} + A^1_{ay} - A^1_{axy}) + (A^1_{bx} + A^1_{by} - A^1_{bxy}) \\
&\qquad - (A^1_{\overline{\overline{ab}}:x} + A^1_{\overline{\overline{ab}}:y} - A^1_{\overline{\overline{ab}}:xy}) \\
&= (A^1_{ax} + A^1_{ay} + A^1_{bx} + A^1_{by}) \\
&\qquad - (A^1_{axy} + A^1_{abx} + A^1_{aby} + A^1_{bxy} + A_{a\overset{1}{b}x} + A_{a\overset{1}{b}y}) \\
&\qquad + (A^1_{abxy} + A_{a\overset{1}{b}xy})
\end{aligned}$$

From observing the method of arriving at this result, any similar complicated benefit of the form $A^{\;\;1}_{\overline{abc\cdots(m)}:\overline{xyz\cdots(n)}}$ may be worked out.

17. To find A^2_{wxyz}

$$\begin{aligned}
A^2_{wxyz} &= A^2_{wx\underset{1}{y}z} + A^2_{wx\underset{1}{y}z} + A^2_{wxy\underset{1}{z}} \\
&= (A^1_{wyz} - A^1_{wxyz}) + (A^1_{wxz} - A^1_{wxyz}) + (A^1_{wxy} - A^1_{wxyz}) \\
&= (A^1_{wyz} + A^1_{wxz} + A^1_{wxy}) - 3A^1_{wxyz}
\end{aligned}$$

18. To find A^{3}_{wxyz}

$$\begin{aligned}
A^{3}_{wxyz} &= A^{2}_{\overset{}{\underset{1}{w(\overline{xy})z}}} + A^{2}_{\underset{1}{wx(\overline{yz})}} + A^{2}_{\underset{1}{w(\overline{xz})y}} \\
&= (A^{1}_{wz} - A^{1}_{wz:\overline{xy}}) + (A^{1}_{wx} - A^{1}_{wx:\overline{yz}}) + (A^{1}_{wy} - A^{1}_{wy:\overline{xz}}) \\
&= \{A^{1}_{wz} - (A^{1}_{wzx} + A^{1}_{wzy} - A^{1}_{wzxy})\} \\
&\qquad + \{A^{1}_{wx} - (A^{1}_{wxy} + A^{1}_{wxz} - A^{1}_{wxyz})\} \\
&\qquad + \{A^{1}_{wy} - (A^{1}_{wyx} + A^{1}_{wyz} - A^{1}_{wyxz})\} \\
&= (A^{1}_{wx} + A^{1}_{wy} + A^{1}_{wz}) - 2(A^{1}_{wxy} + A^{1}_{wxz} + A^{1}_{wyz}) + 3A^{1}_{wxyz}
\end{aligned}$$

19. To find A^{4}_{wxyz}

$$\begin{aligned}
A^{4}_{wxyz} &= A_{w} - A^{1}_{wxyz} - A^{2}_{wxyz} - A^{3}_{wxyz} \\
&= A_{w} - A^{1}_{wxyz} - \{(A^{1}_{wxy} + A^{1}_{wxz} + A^{1}_{wyz}) - 3A^{1}_{wxyz}\} \\
&\qquad - \{(A^{1}_{wx} + A^{1}_{wy} + A^{1}_{wz}) - 2(A^{1}_{wxy} + A^{1}_{wxz} + A^{1}_{wyz}) + 3A^{1}_{wxyz}\} \\
&= A_{w} - (A^{1}_{wx} + A^{1}_{wy} + A^{1}_{wz}) + (A^{1}_{wxy} + A^{1}_{wxz} + A^{1}_{wyz}) - A^{1}_{wxyz}
\end{aligned}$$

$$\begin{aligned}
\text{Or } A^{4}_{wxyz} &= A_{w} - A^{1}_{w:\overline{xyz}} \\
&= A_{w} - (A^{1}_{wx} + A^{1}_{wy} + A^{1}_{wz}) + (A^{1}_{wxy} + A^{1}_{wxz} + A^{1}_{wyz}) - A^{1}_{wxyz}
\end{aligned}$$

20. *Text Book* formula (29) may be easily obtained in a manner similar to that already shown for formula (14) of Chapter X.

$$\bar{a}_{xy} = \int_{0}^{\infty} v^{t}\,{}_{t}p_{xy}\,dt$$

$$\frac{d}{dx}\bar{a}_{xy} = \int_{0}^{\infty} v^{t}\left(\frac{d}{dx}{}_{t}p_{x}\right){}_{t}p_{y}\,dt$$

But as shown on page 212 $\frac{d}{dx}{}_{t}p_{x} = (\mu_{x} - \mu_{x+t})\,{}_{t}p_{x}$

Therefore $\frac{d}{dx}\bar{a}_{xy} = \int_{0}^{\infty} v^{t}(\mu_{x} - \mu_{x+t})\,{}_{t}p_{xy}\,dt$

$$= \mu_{x}\bar{a}_{xy} - \bar{A}^{1}_{xy}$$

And since $\frac{d}{dx}\bar{a}_{xy} = \frac{1}{2}(\bar{a}_{x+1:y} - \bar{a}_{x-1:y})$ approximately

$$\bar{A}^{1}_{xy} = \mu_{x}\bar{a}_{xy} + \frac{1}{2}(\bar{a}_{x-1:y} - \bar{a}_{x+1:y})$$

In the same way, since

$$\bar{a}_{xyz} = \int_0^\infty v^t\,{}_tp_{xyz}\,dt$$

$$\frac{d}{dx}\bar{a}_{xyz} = \int_0^\infty v^t\Big(\frac{d}{dx}{}_tp_x\Big){}_tp_{yz}\,dt$$

$$= \int_0^\infty v^t(\mu_x - \mu_{x+t})\,{}_tp_{xyz}\,dt$$

$$= \mu_x\bar{a}_{xyz} - \bar{A}^1_{xyz}$$

and since $\frac{d}{dx}\bar{a}_{xyz} = \frac{1}{2}(\bar{a}_{x+1:yz} - \bar{a}_{x-1:yz})$ approximately

we have $\bar{A}^1_{xyz} = \mu_x\bar{a}_{xyz} + \frac{1}{2}(\bar{a}_{x-1:yz} - \bar{a}_{x+1:yz})$

which is *Text Book* formula (31).

21. We might obtain the expression for $\bar{A}^1_{xy}$ in *Text Book* formula (40) as follows:—

$$\bar{A}^1_{xy} = \frac{1}{l_x l_y}\int_0^\infty v^t l_{x+t}\,l_{y+t}\,\mu_{x+t}\,dt$$

$$= \frac{c^x}{l_x l_y}\int_0^\infty v^t l_{x+t}\,l_{y+t}\,\mathrm{B}c^t\,dt$$

$$= \frac{c^x}{c^x+c^y}\,\frac{1}{l_x l_y}\int_0^\infty v^t l_{x+t}\,l_{y+t}(\mathrm{B}c^{x+t}+\mathrm{B}c^{y+t})\,dt$$

$$= \frac{c^x}{c^x+c^y}\,\frac{1}{l_x l_y}\int_0^\infty v^t l_{x+t}\,l_{y+t}(\mu_{x+t}+\mu_{y+t})\,dt$$

$$= \frac{c^x}{c^x+c^y}\,\bar{A}_{xy}$$

22. Following, in the case of Gompertz's law, the method adopted by Mr Colenso with regard to Makeham's formula in *J. I. A.*, xxxi. 342, we have this expression:—

$$\bar{A}^1_{xy} = \frac{1}{l_x l_y}\int_0^\infty v^t\,l_{x+t}\,l_{y+t}\,\mathrm{B}c^{x+t}\,dt$$

$$= \frac{\mathrm{B}c^x}{l_x l_y}\int_0^\infty v^t c^t\,l_{x+t}\,l_{y+t}\,dt$$

$= \mu_x\bar{a}'_{xy}$ ($\bar{a}'_{xy}$ being calculated at rate j where $\frac{1}{1+j} = \frac{c}{1+i}$)

$= \mu_x\bar{a}'_w$ ($\bar{a}'_w$ being at the same rate j, and w being such that $\mu_w = \mu_x+\mu_y$).

28. Under Makeham's rule for μ_x, *Text Book* formula (38) may also be obtained as follows :—

$$\begin{aligned}
\bar{A}^1_{xy} &= \frac{1}{l_x l_y}\int_0^\infty v^t l_{x+t}\, l_{y+t}(A + Bc^{x+t})dt \\
&= A\bar{a}_{xy} + \frac{c^x}{l_x l_y}\int_0^\infty v^t l_{x+t}\, l_{y+t}\, Bc^t dt \\
&= A\bar{a}_{xy} + \frac{c^x}{c^x+c^y}\,\frac{1}{l_x l_y}\int_0^\infty v^t l_{x+t}\, l_{y+t}(Bc^{x+t} + Bc^{y+t})dt \\
&= A\bar{a}_{xy} + \frac{c^x}{c^x+c^y}\,\frac{1}{l_x l_y}\int_0^\infty v^t l_{x+t}\, l_{y+t}(\mu_{x+t} + \mu_{y+t} - 2A)dt \\
&= \frac{c^x+c^y}{c^x+c^y}A\bar{a}_{xy} + \frac{c^x}{c^x+c^y}(\bar{A}_{xy} - 2A\bar{a}_{xy}) \\
&= \frac{c^x}{c^x+c^y}\bar{A}_{xy} - \frac{c^x-c^y}{c^x+c^y}A\bar{a}_{xy} \\
&= \frac{c^x}{c^x+c^y}\bar{A}_{xy} + \frac{c^x-c^y}{c^x+c^y}\log s\, \bar{a}_{xy}, \text{ since } A = -\log s
\end{aligned}$$

Also $\bar{A}^1_{xyz\ldots(m)}$

$$\begin{aligned}
&= \frac{1}{l_{xyz\ldots(m)}}\int_0^\infty v^t l_{x+t:y+t:z+t:\ldots(m)}(A + Bc^{x+t})dt \\
&= A\,\bar{a}_{xyz\ldots(m)} + \frac{c^x}{l_{xyz\ldots(m)}}\int_0^\infty v^t l_{x+t:y+t:z+t\ldots(m)}Bc^t dt \\
&= A\bar{a}_{xyz\ldots(m)} + \frac{c^x}{c^x+c^y+c^z+\ldots \text{ to } m \text{ terms}}\,\frac{1}{l_{xyz\ldots(m)}}\int_0^\infty v^t l_{x+t:y+t:z+t\ldots(m)} \\
&\qquad \times (Bc^{x+t} + Bc^{y+t} + Bc^{z+t} + \ldots \text{ to } m \text{ terms})dt \\
&= A\bar{a}_{xyz\ldots(m)} + \frac{c^x}{c^x+c^y+c^z+\ldots \text{ to } m \text{ terms}}\,\frac{1}{l_{xyz\ldots(m)}}\int_0^\infty v^t l_{x+t:y+t:z+t\ldots(m)} \\
&\qquad \times (\mu_{x+t} + \mu_{y+t} + \mu_{z+t} + \ldots \text{ to } m \text{ terms} - mA)dt \\
&= A\bar{a}_{xyz\ldots(m)} + \frac{c^x}{c^x+c^y+c^z+\ldots \text{ to } m \text{ terms}}(\bar{A}_{xyz\ldots(m)} - mA\bar{a}_{xyz\ldots(m)}) \\
&= \frac{c^x}{c^x+c^y+c^z+\ldots \text{ to } m \text{ terms}}\bar{A}_{xyz\ldots(m)} \\
&\qquad - \frac{mc^x - (c^x+c^y+c^z+\ldots \text{ to } m \text{ terms})}{c^x+c^y+c^z+\ldots \text{ to } m \text{ terms}}A\bar{a}_{xyz\ldots(m)}
\end{aligned}$$

(putting $c^x+c^y+c^z+\cdots$ to m terms $=mc^w$, and $\bar{a}_{xyz\ldots(m)}=\bar{a}_{www\ldots(m)}$)

$$= \frac{c^x}{mc^w}\bar{\mathrm{A}}_{www\ldots(m)} + \left(\frac{c^x}{c^w} - 1\right)\log s\,\bar{a}_{www\ldots(m)}$$

24. Mr Colenso in the paper already mentioned deduces the following expressions:—

$$\bar{\mathrm{A}}^1_{xy} = \frac{1}{l_x l_y}\int_0^\infty v^t l_{x+t}\,l_{y+t}(\mathrm{A}+\mathrm{B}c^{x+t})dt$$

$$= \mathrm{A}\bar{a}_{xy} + \frac{\mathrm{B}c^x}{l_x l_y}\int_0^\infty v^t c^t l_{x+t}\,l_{y+t}dt$$

$$= \mathrm{A}\bar{a}_{xy} + (\mu_x - \mathrm{A})\bar{a}'_{xy}$$

$\left(\bar{a}'_{xy}\right.$ being calculated at rate j where $\left.\dfrac{1}{1+j} = \dfrac{c}{1+i}\right)$

$$= \mathrm{A}\bar{a}_{ww} + (\mu_x - \mathrm{A})\bar{a}'_{ww}$$

($\bar{a}'_{ww}$ being at the same rate j, and w such that $\mu_x+\mu_y=2\mu_w$)

$$= -\log s\,\bar{a}_{ww} + (\mu_x + \log s)\bar{a}'_{ww} \quad \text{since } \mathrm{A} = -\log s.$$

Similarly

$$\bar{\mathrm{A}}^1_{xyz} = -\log s\,\bar{a}_{www} + (\mu_x + \log s)\bar{a}'_{www}$$

And generally

$$\bar{\mathrm{A}}^1_{xyz\ldots(m)} = -\log s\,\bar{a}_{www\ldots(m)} + (\mu_x + \log s)\bar{a}'_{www\ldots(m)}$$

Mr Colenso gives tables of $-\log_e s\,\bar{a}_{www}$, $\log_{10}(\mu_x+\log_e s)$, and $\log_{10}\bar{a}'_{www}$ from which values of $\bar{\mathrm{A}}^1_{xyz}$ may be easily calculated. Basis: Carlisle Table of Mortality, rate of interest 3 per cent.

25. *Text Book* formula (41) may be obtained thus:—

$$\bar{A}_{\overline{xyz\ldots(m)}:abc\ldots(n)}^{\;1}$$

$$= \frac{1}{l_{xyz\ldots(m).\,abc\ldots(n)}}\int_0^\infty v^t l_{x+t:y+t:z+t\ldots(m)} l_{a+t:b+t:c+t\ldots(n)}$$

$$\times\{mA + B(c^{x+t} + c^{y+t} + c^{z+t} + \ldots \text{ to } m \text{ terms})\}dt$$

$$= mA\bar{a}_{xyz\ldots(m).\,abc\ldots(n)} + \frac{c^x + c^y + c^z + \ldots \text{ to } m \text{ terms}}{c^x + c^y + c^z + \ldots \text{ to } m \text{ terms} + c^a + c^b + \ldots \text{ to } n \text{ terms}}$$

$$\times \frac{1}{l_{xyz\ldots(m).\,abc\ldots(n)}}\int_0^\infty v^t l_{x+t:y+t:z+t\ldots(m)} l_{a+t:b+t:c+t\ldots(n)}$$

$$\times \{\mu_{x+t} + \mu_{y+t} + \mu_{z+t} + \ldots \text{ to } m \text{ terms}$$

$$+ \mu_{a+t} + \mu_{b+t} + \mu_{c+t} + \ldots \text{ to } n \text{ terms } - (m+n)A\}dt$$

$$= \frac{c^x + c^y + c^z + \ldots \text{ to } m \text{ terms}}{c^x + c^y + c^z + \ldots + c^a + c^b + \ldots \text{ to } (m+n) \text{ terms}}\bar{A}_{xyzabc\ldots(m+n)}$$

$$+ \frac{n(c^x + c^y + c^z + \ldots \text{ to } m \text{ terms}) - m(c^a + c^b + \ldots \text{ to } n \text{ terms})}{c^x + c^y + c^z + \ldots + c^a + c^b + \ldots \text{ to } (m+n) \text{ terms}}$$

$$\times \log s\,\bar{a}_{xyzabc\ldots(m+n)}$$

since $A = -\log s$.

26. *Text Book* formula (42) may be obtained directly in a manner which throws light on the ordinary assurance of the same kind proved in *Text Book,* Article 21.

The benefit may be divided up as follows:—(1) An assurance of 1 payable at the moment of death of (x) provided (y) be then alive, and (2) a temporary assurance for t years to be entered on by (x) at the death of (y). Thus

$$\bar{A}^1_{x:\overline{y(\overline{t|})}} = \bar{A}^1_{xy} + \frac{1}{l_x l_y}\int_0^\infty v^n l_{x+n}\, l_{y+n}\mu_{y+n}\bar{A}^{\;1}_{\overline{x+n}:\overline{t|}}dn$$

$$= \bar{A}^1_{xy} + \frac{1}{l_x l_y}\int_0^\infty v^n l_{x+n}\, l_{y+n}\mu_{y+n}(\bar{A}_{x+n} - v^t{}_tp_{x+n}\bar{A}_{x+n+t})dn$$

$$= \bar{A}^1_{xy} + \bar{A}^2_{xy} - \frac{v^t{}_tp_x}{l_{x+t}l_y}\int_0^\infty v^n l_{x+t+n}\, l_{y+n}\mu_{y+n}\bar{A}_{x+t+n}dn$$

$$= \bar{A}_x - v^t{}_tp_x\bar{A}^{\;2}_{\overline{x+t}:y}$$

$$= \bar{A}_x - \frac{D_{x+t}}{D_x}(\bar{A}_{x+t} - \bar{A}^{\;1}_{\overline{x+t}:y})$$

EXAMPLES

1. Find an expression for A^{1}_{xy} on the assumption that the chance of (x) and (y) dying in the same year may be neglected.

For the complete value of A^{1}_{xy} we have the formula

$$A^{1}_{xy} = \Sigma v^n\{({}_{n-1}p_x - {}_np_x){}_np_y + \tfrac{1}{2}({}_{n-1}p_x - {}_np_x)({}_{n-1}p_y - {}_np_y)\}$$

But as the chance of both deaths occurring in the same year may be neglected, we omit the second term in the expression, and we have for the value of A^{1}_{xy} under the conditions specified

$$\begin{aligned}\Sigma v^n&({}_{n-1}p_x - {}_np_x){}_np_y \\ &= \Sigma v^n\left(\frac{{}_np_{x-1:y}}{p_{x-1}} - {}_np_{xy}\right) \\ &= \frac{a_{x-1:y}}{p_{x-1}} - a_{xy}\end{aligned}$$

This benefit is of use when studying formula (11) of *Text Book*, Chapter XIV., and its modification in formula (14).

2. Given the values of single- and joint-life annuities, find the annual premium payable during the joint lives of (x) and (y) for an assurance payable on the death of the last survivor of (x) and (y), but half the sum assured to be payable on each death if (x) dies before (y).

Do you see any objection to making the premium payable during the life of the last survivor?

The benefit splits into two parts: where (x) is the survivor the assurance is payable on his death, and where (y) is the survivor half is payable on each death. Therefore the whole value is

$$\begin{aligned}A^{2}_{xy} &+ \tfrac{1}{2}A^{1}_{xy} + \tfrac{1}{2}A_{x\overset{2}{y}} \\ &= (A_x - A^{1}_{xy}) + \tfrac{1}{2}A^{1}_{xy} + \tfrac{1}{2}(A_y - A_{x\overset{1}{y}}) \\ &= A_x + \tfrac{1}{2}A_y - \tfrac{1}{2}A_{xy} \\ &= 1 - d(1 + a_x) + \tfrac{1}{2}\{1 - d(1 + a_y) - 1 + d(1 + a_{xy})\} \\ &= 1 - d(1 + a_x + \tfrac{1}{2}a_y - \tfrac{1}{2}a_{xy})\end{aligned}$$

As the premium is to be payable throughout the joint lives the payment side

$$= \mathrm{P}(1+a_{xy})$$

and
$$\mathrm{P} = \frac{1-d(1+a_x+\frac{1}{2}a_y-\frac{1}{2}a_{xy})}{1+a_{xy}}$$

The objection to making the premium payable throughout the life of the last survivor is that, if (x) were to die in the early years of the contract, then (y) might be able to secure the benefit of $\frac{1}{2}$ payable on his own death at a smaller premium, provided he were in good health.

3. Deduce a formula for the annual premium for an assurance payable if a life aged x dies within the next five years, or if he lives five years and dies after another life now aged y.

The first part of the benefit is $\mathrm{A}^1_{x:\overline{5}|}$; for the second part, if (y) also lives the five years, we have $v^5{}_5p_{xy}\mathrm{A}^{\;\;2}_{\overline{x+5}:\overline{y+5}}$, and if (y) dies within five years, $v^5{}_5p_x(1-{}_5p_y)\mathrm{A}_{x+5}$. Therefore the whole benefit is

$$\begin{aligned}
&\mathrm{A}^1_{x:\overline{5}|}+v^5{}_5p_{xy}\mathrm{A}^{\;\;2}_{\overline{x+5}:\overline{y+5}}+v^5{}_5p_x(1-{}_5p_y)\mathrm{A}_{x+5}\\
&= \mathrm{A}_x-v^5{}_5p_x\mathrm{A}_{x+5}+v^5{}_5p_{xy}(\mathrm{A}_{x+5}-\mathrm{A}^{\;\;1}_{\overline{x+5}:\overline{y+5}})\\
&\qquad +v^5{}_5p_x\mathrm{A}_{x+5}-v^5{}_5p_{xy}\mathrm{A}_{x+5}\\
&= \mathrm{A}_x-v^5{}_5p_{xy}\mathrm{A}^{\;\;1}_{\overline{x+5}:\overline{y+5}}
\end{aligned}$$

Similarly the payment side is

$$\begin{aligned}
&\mathrm{P}\{\mathrm{a}_{x:\overline{5}|}+v^5{}_5p_{xy}\mathrm{a}_{x+5}+v^5{}_5p_x(1-{}_5p_y)\mathrm{a}_{x+5}\}\\
&= \mathrm{P}(\mathrm{a}_x-v^5{}_5p_x\mathrm{a}_{x+5}+v^5{}_5p_{xy}\mathrm{a}_{x+5}+v^5{}_5p_x\mathrm{a}_{x+5}-v^5{}_5p_{xy}\mathrm{a}_{x+5})\\
&= \mathrm{P}\mathrm{a}_x
\end{aligned}$$

Hence
$$\mathrm{P} = \frac{\mathrm{A}_x-v^5{}_5p_{xy}\mathrm{A}^{\;\;1}_{\overline{x+5}:\overline{y+5}}}{\mathrm{a}_x}$$

4. Express the value of an annuity-certain for n years, payable quarterly, to begin to run at the death of (x) if he die after (y).

On the required contingency happening the value of the annuity is $\frac{1}{4}a_{\overline{4n}|}$ calculated at rate of interest $\frac{i}{4}$. Therefore the value of the annuity at the present time is

$$\overline{A}^2_{xy} \times \tfrac{1}{4}a_{\overline{4n}|}\left(\tfrac{i}{4}\right) = \tfrac{1}{4}(1+i)^{\frac{1}{2}}(A_x - A^1_{xy})a_{\overline{4n}|}\left(\tfrac{i}{4}\right)$$

5. Deduce a formula for the annual premium for an assurance payable on the death of (x) if (y) has died five years or more before him, the premium to be payable during the currency of the assurance.

The benefit here is equivalent to an assurance on (x), less an assurance payable if he dies before (y) or within five years after (y), that is

$$A_x - A^1_{x:\overline{y(\overline{5}|)}}$$

The premium will be paid throughout the whole of (x)'s life, and the payment side is equal to

$$P(1+a_x)$$

whence $$P = \frac{A_x - A^1_{x:\overline{y(\overline{5}|)}}}{1+a_x}$$

6. State a formula for the annual premium for an insurance payable t years after the death of (x), if (y) has survived him and died before the end of the t years.

Benefit side

$$= \Sigma v^{n-\frac{1}{2}} \frac{d_{x+n-1}}{l_x} \frac{l_{y+n-\frac{1}{2}}}{l_y} v^t(1 - {}_tp_{y+n-\frac{1}{2}})$$

$$= v^t\left(\Sigma v^{n-\frac{1}{2}} \frac{d_{x+n-1}}{l_x} \frac{l_{y+n-\frac{1}{2}}}{l_y} - \Sigma v^{n-\frac{1}{2}} \frac{d_{x+n-1}}{l_x} \frac{l_{y+n+t-\frac{1}{2}}}{l_y}\right)$$

$$= v^t(\overline{A}^1_{xy} - {}_tp_y\overline{A}^1_{x:\overline{y+t}})$$

The premium will be payable so long as (y) survives jointly with (x) and for t years after (x)'s death, if (y) lives so long.

Therefore payment side $= P(1 + a_{y:\overline{x(\overline{t}|)}})$

Equating the two sides we have

$$P = \frac{v^t(\overline{A}^1_{xy} - {}_tp_y\overline{A}^1_{x:\overline{y+t}})}{1+a_{y:\overline{x(\overline{t}|)}}}$$

7. Give the formula for the single premium for an assurance payable at the death of the last survivor of (x) and (y), if that occur in the lifetime of (z), or within t years after the death of (z).

$$\begin{aligned}
A_{\overset{1}{\overline{xy}}:\overline{z(\overline{t|})}} &= A^{1}_{x:\overline{z(\overline{t|})}} + A^{1}_{y:\overline{z(\overline{t|})}} - A_{\overset{1}{\overline{xy}}:\overline{z(\overline{t|})}} \\
&= \left\{A_x - \frac{D_{x+t}}{D_x}\left(A_{x+t} - A_{\overset{1}{x+t}:z}\right)\right\} \\
&\quad + \left\{A_y - \frac{D_{y+t}}{D_y}\left(A_{y+t} - A_{\overset{1}{y+t}:z}\right)\right\} \\
&\quad - \left\{A_{xy} - \frac{D_{x+t:y+t}}{D_{xy}}\left(A_{x+t:y+t} - A_{\overset{1}{x+t}:\overline{y+t}:z} - A_{\overline{x+t}:\overset{1}{y+t}:z}\right)\right\}
\end{aligned}$$

8. Find the annual premium for an assurance payable at the death of (x), unless (y) die within the first n years and in the lifetime of (x).

To get the benefit we must deduct from the ordinary assurance on (x) the value (1) of an assurance payable should (x) die after (y) within n years, and (2) of an assurance payable should (x) die after n years, (y) having died within n years.

The benefit side is

$$\begin{aligned}
& A_x - \{|_n A^{2}_{xy} + (1 - {}_np_y)\,{}_n|A_x\} \\
&= A_x - |_n A_x + |_n A^{1}_{xy} - {}_n|A_x + v^n\,{}_np_{xy}\,A_{x+n} \\
&= |_n A^{1}_{xy} + v^n\,{}_np_{xy}\,A_{x+n}
\end{aligned}$$

which is correct, being the assurance payable should (x) die before (y) within n years, or should (x) die after n years, (y) having survived that period.

The payment side is

$$P(1 + a_{xy:\overline{n-1|}} + v^n\,{}_np_{xy}\,a_{x+n})$$

$$\text{And } P = \frac{|_n A^{1}_{xy} + v^n\,{}_np_{xy}\,A_{x+n}}{1 + a_{xy:\overline{n-1|}} + v^n\,{}_np_{xy}\,a_{x+n}}$$

9. Find the annual premium for an assurance on the life of (x), the policy money to be payable at death or on the expiry of n years, provided that in either case two other lives (y) and (z) are then in existence.

The benefit side is

$$|_n A^1_{xyz} + v^n {}_n p_{xyz}$$

And the payment side is

$$P(1 + a_{xyz:\overline{n-1}|})$$

Hence $$P = \frac{|_n A^1_{xyz} + v^n {}_n p_{xyz}}{1 + a_{xyz:\overline{n-1}|}}$$

10. Give the formula for the annual premium for a temporary assurance of 1 payable in the event of (x) dying before (y) within n years, (z) having died previously.

The benefit side is

$$|_n A^2_{\underset{1}{xyz}} = |_n A^1_{xy} - |_n A^1_{xyz}$$

The assurance will not cease on the death of (z), and therefore that life does not come into account in settling the currency of the payment side, which accordingly is

$$P(1 + a_{xy:\overline{n-1}|})$$

$$\text{And } P = \frac{|_n A^1_{xy} - |_n A^1_{xyz}}{1 + a_{xy:\overline{n-1}|}}$$

11. Deduce the annual premium for an assurance payable on the death of (x) if he attains age $x+n$ and dies before (y) and after (z), the premium to be payable throughout the whole period of the status.

The benefit side $= {}_{n|}A^1_{xy} - {}_{n|}A^1_{xyz}$

As in the previous question the death of (z) will not disturb the continuance of the assurance, and therefore the premium will be payable so long as (x) and (y) jointly survive. The payment side is accordingly

$$P(1 + a_{xy})$$

$$\text{and } P = \frac{{}_{n|}A^1_{xy} - {}_{n|}A^1_{xyz}}{1 + a_{xy}}$$

12. How would you arrive at the annual premium for an assurance payable at the death of the last survivor of three

lives aged 40, 50, and 60 respectively, provided a life aged 20 is dead before the happening of the death of the survivor?

$$\text{The benefit side} \;=\; A_{20:\overline{40:50:60}}^{\;\;\;2} = A_{\overline{40:50:60}} - A^{\;\;\;1}_{\overline{40:50:60}:20}$$

$$\text{The payment side} \;=\; P(1 + a_{\overline{40:50:60}})$$

$$\text{Hence}\quad P = \frac{A_{\overline{40:50:60}} - A^{\;\;\;1}_{\overline{40:50:60}:20}}{1 + a_{\overline{40:50:60}}}$$

$$\text{and}\quad A^{\;\;\;1}_{\overline{40:50:60}:20} = A_{\overset{1}{40}:20} + A_{\overset{1}{50}:20} + A_{\overset{1}{60}:20} - A_{\overset{1}{40}:50:20}$$

$$- A_{40:\overset{1}{50}:20} - A_{\overset{1}{40}:60:20} - A_{40:\overset{1}{60}:20} - A_{\overset{1}{50}:60:20}$$

$$- A_{50:\overset{1}{60}:20} + A_{\overset{1}{40}:50:60:20} + A_{40:\overset{1}{50}:60:20} + A_{40:50:\overset{1}{60}:20}$$

13. A sum of 1 is to be divided among such of the existing children of a widow aged w as may be alive at her death. What is the share of (x), (a) assuming that there are two children aged x and y respectively now alive, (b) assuming that there are three now alive aged x, y, and z respectively?

(a) If both (x) and (y) are alive at (w)'s death then (x) receives $\frac{1}{2}$, but if (y) has died previously (x) receives 1. Therefore the value of (x)'s share is

$$\tfrac{1}{2}A_{\overset{1}{w}xy} + A_{\overset{2}{w}x\underset{1}{y}}$$

$$= \tfrac{1}{2}A_{\overset{1}{w}xy} + (A_{\overset{1}{w}x} - A_{\overset{1}{w}xy})$$

$$= A_{\overset{1}{w}x} - \tfrac{1}{2}A_{\overset{1}{w}xy}$$

(b) If all three are alive at death of (w), (x) receives $\frac{1}{3}$; if (x) and one other only are then alive, he receives $\frac{1}{2}$; and if he alone is alive at death of (w), he receives 1. Therefore his share is

$$\tfrac{1}{3}A_{\overset{1}{w}xyz} + \tfrac{1}{2}(A_{\overset{2}{w}x\underset{1}{y}z} + A_{\overset{2}{w}xy\underset{1}{z}}) + (A_{\overset{3}{w}x\underset{21}{yz}} + A_{\overset{3}{w}x\underset{12}{yz}})$$

$$= \tfrac{1}{3}A_{\overset{1}{w}xyz} + \tfrac{1}{2}(A_{\overset{1}{w}xz} - A_{\overset{1}{w}xyz} + A_{\overset{1}{w}xy} - A_{\overset{1}{w}xyz})$$

$$+ (A_{\overset{1}{w}x} - A_{\overset{1}{w}xy} - A_{\overset{1}{w}xz} + A_{\overset{1}{w}xyz})$$

$$= A_{\overset{1}{w}x} - \tfrac{1}{2}(A_{\overset{1}{w}xy} + A_{\overset{1}{w}xz}) + \tfrac{1}{3}A_{\overset{1}{w}xyz}$$

14. Given four lives (x), (a), (b), and (c), find the value of an assurance to yield at (x)'s death £1000 if one and only one of the lives (a), (b), and (c) shall have predeceased him, and £3000 if

two and only two shall have predeceased him. The expression is to be reduced to assurances which determine on the first death.

The value of the assurance is

$$
\begin{aligned}
&1000\mathrm{A}^{2}_{xabc}+3000\mathrm{A}^{3}_{xabc}\\
&\quad= 1000(\mathrm{A}^{1}_{xab}+\mathrm{A}^{1}_{xac}+\mathrm{A}^{1}_{xbc}-3\mathrm{A}^{1}_{xabc})\\
&\qquad\qquad +3000(\mathrm{A}^{1}_{xa}+\mathrm{A}^{1}_{xb}+\mathrm{A}^{1}_{xc}-2\mathrm{A}^{1}_{xab}-2\mathrm{A}^{1}_{xac}-2\mathrm{A}^{1}_{xbc}+3\mathrm{A}^{1}_{xabc})\\
&\quad= 3000(\mathrm{A}^{1}_{xa}+\mathrm{A}^{1}_{xb}+\mathrm{A}^{1}_{xc})-5000(\mathrm{A}^{1}_{xab}+\mathrm{A}^{1}_{xac}+\mathrm{A}^{1}_{xbc})+6000\mathrm{A}^{1}_{xabc}
\end{aligned}
$$

15. Three partners, A, B, and C, aged respectively 30, 35, and 40, possess a capital of £10,000, and their proportionate interests in the business are 2, 3, and 5 tenths. How would you calculate the premium for an assurance to cover the risk of having to pay out the representatives of the partners who may happen to die first and second?

The value of the assurance required is

$$2000\mathrm{A}^{1}_{30:\overline{35:40}}+3000\mathrm{A}^{1}_{35:\overline{30:40}}+5000\mathrm{A}^{1}_{40:\overline{30:35}}$$

It would be advisable that three separate policies should be effected, either by single or annual premiums, one for each part of the above benefit. If, however, one annual-premium policy is essential, the payment side will be

$$\mathrm{P}(1+a_{\overline{30:35:40}}^{\;2}) = \mathrm{P}(1+a_{30:35}+a_{30:40}+a_{35:40}-2a_{30:35:40})$$

$$\text{Hence}\qquad \mathrm{P} = 1000\,\frac{2\mathrm{A}^{1}_{30:\overline{35:40}}+3\mathrm{A}^{1}_{35:\overline{30:40}}+5\mathrm{A}^{1}_{40:\overline{30:35}}}{1+a_{30:35}+a_{30:40}+a_{35:40}-2a_{30:35:40}}$$

It is possible that under such a policy an option may be exercised against the office in the event of one of the lives dying early, say C in the first year. The premium for the remainder of the benefit, viz. $\dfrac{2000\mathrm{A}^{1}_{31:36}+3000\mathrm{A}^{1}_{36:31}}{1+a_{31:36}}$, will probably work out at less than P as found above.

16. Give a formula for $\mathrm{P}^{1}_{30:\overline{10|}:\overline{60}}$, the annual premium for an assurance on (30) payable in the event of his dying within 10 years or before (60).

The benefit side is

$$A^{\,1}_{30:\overline{\overline{10|:60}}} = A^{\,1}_{30:\overline{10|}} + A^{\,1}_{30:60} - A^{\,1}_{30:60:\overline{10|}}$$

The premium will be payable so long as (30) survives jointly with the survivor of 10 years certain and of (60), and for the payment side we have

$$P^{\,1}_{30:\overline{\overline{10|:60}}} \times a_{30:\overline{\overline{10|:60}}}$$

$$= P^{\,1}_{30:\overline{\overline{10|:60}}}(a_{30:\overline{10|}} + a_{30:60} - a_{30:60:\overline{10|}})$$

Therefore $$P^{\,1}_{30:\overline{\overline{10|:60}}} = \frac{A^{\,1}_{30:\overline{10|}} + A^{\,1}_{30:60} - A^{\,1}_{30:60:\overline{10|}}}{a_{30:\overline{10|}} + a_{30:60} - a_{30:60:\overline{10|}}}$$

With the premium payable during a status such as this, it is possible for an option to be exercised against the insuring office in the event of (60) dying within 10 years, say at the end of the tth year. For at that date, provided (30) is in good health, he may obtain an equivalent benefit for $P_{\overset{1}{30+t}:\overline{10-t|}}$ which might be less than the premium found as above. On the other hand, it would not do to make the premium payable so long as (30), (60), and 10 years certain survive jointly, unless evidence as to the health of (60) were produced. For if (60) were dying, then (30) would secure a short-term assurance for 10 years for a very inadequate single premium.

17. Use Mr Colenso's tables (*J. I. A.*, xxxi. 354-6) to find the value of $\bar{A}^{\,1}_{85:72:79}$ and ${}^{(\infty)}P^{\,1}_{85:72:79}$

$$\begin{aligned} \bar{A}^{\,1}_{85:72:79} &= -\log s\,\bar{a}_{85:72:79} + (\mu_{85} + \log s)\bar{a}'_{85:72:79} \\ &= -\log s\,\bar{a}_{71\cdot73:71\cdot73:71\cdot73} + (\mu_{85} + \log s)\bar{a}'_{71\cdot73:71\cdot73:71\cdot73} \end{aligned}$$

since $$\begin{aligned} \mu_{85} + \mu_{72} + \mu_{79} &= \cdot01020 + \cdot06558 + \cdot11692 \\ &= \cdot19270 \\ &= 3\mu_{71\cdot73} \end{aligned}$$

Now $$-\log s\,\bar{a}_{71\cdot73:71\cdot73:71\cdot73} = \cdot02930$$

and $$\begin{aligned} \log_{10}(\mu_{85} + \log s) &= \bar{3}\cdot29072 \\ \log_{10}\bar{a}'_{71\cdot73:71\cdot73:71\cdot73} &= \cdot68205 \\ \hline & \bar{3}\cdot97277 = \log_{10}\cdot00939 \end{aligned}$$

Therefore $\bar{A}^{1}_{85:72:79} = \cdot 02930 + \cdot 00939$

$= \cdot 03869$, or £3, 17s. 5d. % very nearly.

$$^{(\infty)}P^{1}_{85:72:79} = \frac{\bar{A}^{1}_{85:72:79}}{\frac{1}{2} + \bar{a}_{85:72:79}} \text{ approximately.}$$

$$= \frac{\cdot 03869}{4 \cdot 053}$$

$= \cdot 00955$, or 19s. 1d. % very nearly.

18. Investigate an expression for $\frac{d}{dx}\bar{a}_{x\overline{n}|}$, and show what approximate conclusion this leads to on the assumption that Makeham's law holds.

$$\frac{d}{dx}\bar{a}_{x\overline{n}|} = \frac{d}{dx}\int_0^n v^t {}_tp_x dt$$

$$= \int_0^n v^t \frac{d}{dx}\left(\frac{l_{x+t}}{l_x}\right)dt$$

$$= -\int_0^n v^t(\mu_{x+t} - \mu_x)\frac{l_{x+t}}{l_x}dt$$

$$= \mu_x \int_0^n v^t {}_tp_x dt - \int_0^n v^t {}_tp_x \mu_{x+t} dt$$

$$= \mu_x \bar{a}_{x\overline{n}|} - \bar{A}^{1}_{x\overline{n}|}$$

On Makeham's hypothesis $\mu_{x+t} - \mu_x = Bc^x(c^t - 1)$, and therefore

$$\frac{d}{dx}\bar{a}_{x\overline{n}|} = -Bc^x \int_0^n v^t(c^t - 1){}_tp_x dt$$

Considering the definite integral, we see that it represents either

(*a*) The value of a temporary annuity with increasing payments; or

(*b*) The difference between the values of two uniform annuities, one calculated at the ordinary rate of interest, say i, the other calculated at rate j such that $\frac{1}{1+j} = \frac{c}{1+i}$

19. Show that on Makeham's hypothesis $\bar{A}^{1}_{xyz} = \dfrac{\log s}{3\log s - \delta}$ for all values of x, y, and z, which satisfy the equation $\dfrac{\delta}{\log s} = \dfrac{2\mu_x - \mu_y - \mu_z}{\mu_x + \log s}$

$$\text{Since} \qquad \frac{\delta}{\log s} = \frac{2\mu_x - \mu_y - \mu_z}{\mu_x + \log s}$$

$$= \frac{B(2c^x - c^y - c^z)}{Bc^x}$$

$$= \frac{2c^x - c^y - c^z}{c^x}$$

$$\text{Therefore} \qquad \delta c^x = \log s(2c^x - c^y - c^z)$$

$$= \log s\{3c^x - (c^x + c^y + c^z)\}$$

$$c^x + c^y + c^z = \frac{c^x(3\log s - \delta)}{\log s}$$

But by *Text Book* formula (41)

$$\bar{A}^{1}_{xyz} = \frac{c^x}{c^x + c^y + c^z}\bar{A}_{xyz} - \frac{c^y + c^z - 2c^x}{c^x + c^y + c^z}\log s\, \bar{a}_{xyz}$$

substituting $1 - \delta\bar{a}_{xyz}$ for $\bar{A}_{xyz}$

$$= \frac{c^x}{c^x + c^y + c^z} - \frac{c^x}{c^x + c^y + c^z}\delta\bar{a}_{xyz} - \frac{c^y + c^z - 2c^x}{c^x + c^y + c^z}\log s\, \bar{a}_{xyz}$$

substituting for δc^x its value as found above

$$= \frac{c^x}{c^x + c^y + c^z}$$

$$= c^x \frac{\log s}{c^x(3\log s - \delta)}$$

$$= \frac{\log s}{3\log s - \delta}$$

20. Find an expression for $\dfrac{d}{di}\,\bar{a}_{xy}$.

$$\begin{aligned}\frac{d}{di}\,\bar{a}_{xy} &= \frac{dv}{di}\frac{d}{dv}\,\bar{a}_{xy}\\ &= \frac{dv}{di}\int_0^\infty \frac{d}{dv}\,v^t{}_tp_{xy}dt\\ &= -v^2\int_0^\infty tv^{t-1}{}_tp_{xy}dt\\ &= -v\int_0^\infty tv^t{}_tp_{xy}dt\\ &= -v(\mathrm{I}\bar{a})_{xy}\end{aligned}$$

21. Give a formula for the fine, to be paid as a single premium at the outset, for the option to increase at the end of n years, without further inquiry as to health, the sum assured by a survivorship policy payable only in the event of (x) dying before (y).

If, on the option being exercised at the end of n years, the premium to be paid is to remain at the rate of $\mathrm{P}_{\overset{1}{[x]}[y]}$ per unit for the future, the difference between that premium and the premium, which, looking to the effect of selection, should be charged, $(\mathrm{P}_{\overset{1}{\overline{[x]+n}}:\overline{[y]+n}})$, is $(\mathrm{P}_{\overset{1}{\overline{[x]+n}}:\overline{[y]+n}} - \mathrm{P}_{\overset{1}{[x]}[y]})$. The whole value of this difference for the period after n years is then

$$\frac{\mathrm{D}_{[x]+n:[y]+n}}{\mathrm{D}_{[x][y]}}(\mathrm{P}_{\overset{1}{\overline{[x]+n}}:\overline{[y]+n}} - \mathrm{P}_{\overset{1}{[x]}[y]})(1 + a_{[x]+n:[y]+n}).$$

If, however, when the option is to be exercised, the premium to be paid is that for a similar benefit at the then ages, we must substitute for $\mathrm{P}_{\overset{1}{[x]}[y]}$ in the above formula the premium $\mathrm{P}_{\overset{1}{\overline{[x+n]}}\,\overline{[y+n]}}$.

22. A select life, (x), desires a contingent insurance against (y) with the proviso that, if he be alive at the death of (y), he shall have the option of converting his policy, as at the next renewal date, into a whole-life assurance at the ordinary annual premium applicable to his then age without medical examination. Obtain an expression for the net annual charge required for this option.

If the option is assumed to be exercised at the end of the tth year its value is

$$(\mathrm{P}_{[x]+t} - \mathrm{P}_{[x+t]})a_{[x]+t}$$

and the probability of its having to be exercised then is

$$_{t-1|}q_{[y]}\,{}_tp_{[x]}'$$

Therefore the value at present of the option in respect of the tth year is

$$v^t{}_{t-1|}q_{[y]}\,{}_tp_{[x]}(\mathrm{P}_{[x]+t}-\mathrm{P}_{[x+t]})\mathrm{a}_{[x]+t}$$

Summing this for every value of t and dividing by $\mathrm{a}_{[x][y]}$ we obtain the addition to the ordinary premium to cover the option

$$\frac{\Sigma v^t{}_{t-1|}q_{[y]}\,{}_tp_{[x]}(\mathrm{P}_{[x]+t}-\mathrm{P}_{[x+t]})\mathrm{a}_{[x]+t}}{\mathrm{a}_{[x][y]}}$$

$$=\frac{\Sigma_{t-1|}q_{[y]}(\mathrm{P}_{[x]+t}-\mathrm{P}_{[x+t]})(\mathrm{a}_{[x]}-\mathrm{a}_{[x]\overline{t|}})}{\mathrm{a}_{[x][y]}}$$

23. Show how to find approximately the net annual premium, on the basis of the Makeham graduation of the Carlisle Table, at 3 per cent. interest, for an assurance payable in the event of (x) predeceasing (y), (1) (x) only, and (2) both (x) and (y), being resident in a foreign country. It may be assumed that the extra risk is represented in the case of a single life by a constant addition of ·01 to the force of mortality at all ages.

$$(1)\qquad {}^{(\infty)}\mathrm{P}'^{\,1}_{xy}=\frac{\overline{\mathrm{A}}'^{\,1}_{xy}}{\frac{1}{2}+\bar{a}'_{xy}}$$

where

$$\overline{\mathrm{A}}'^{\,1}_{xy}=\int_0^\infty v^t{}_tp'_x\,{}_tp_y\mu'_{x+t}\,dt$$

$$=\int_0^\infty v^tr^t{}_tp_x\,{}_tp_y(\mu_{x+t}+\cdot01)dt$$

$$=\overline{\mathrm{A}}'^{\,1}_{xy}+\cdot01\bar{a}'_{xy}$$

both $\overline{\mathrm{A}}'^{\,1}_{xy}$ and $\bar{a}'_{xy}$ being calculated at a special rate of interest j such that $\dfrac{1}{1+j}=\dfrac{r}{1\cdot03}$, where $-\log r=\cdot01$.

Also

$$\bar{a}'_{xy}=\int_0^\infty v^t{}_tp'_x\,{}_tp_y dt$$

$$=\int_0^\infty v^tr^t{}_tp_x\,{}_tp_y dt$$

$$=\bar{a}'_{xy}\text{ found as above.}$$

(2) $$^{(\infty)}\mathrm{P}''^{1}_{xy} = \frac{\bar{\mathrm{A}}''^{1}_{xy}}{\frac{1}{2}+\bar{a}''_{xy}}$$

where $$\bar{\mathrm{A}}''^{1}_{xy} = \int_0^\infty v^t{}_tp'_x{}_tp'_y\mu'_{x+t}dt$$

$$= \int_0^\infty v^tr^{2t}{}_tp_{xy}(\mu_{x+t}+\cdot01)dt$$

$$= \bar{\mathrm{A}}''^{1}_{xy}+\cdot01\bar{a}''_{xy}$$

both $\bar{\mathrm{A}}''^{1}_{xy}$ and $\bar{a}''_{xy}$ being calculated at rate j which is such that $\frac{1}{1+j}=\frac{r^2}{1\cdot03}$, where $-\log r = \cdot01$.

Also $$\bar{a}''_{xy} = \int_0^\infty v^t{}_tp'_x{}_tp'_ydt$$

$$= \int_0^\infty v^tr^{2t}{}_tp_{xy}dt$$

$$= \bar{a}''_{xy} \text{ as above.}$$

CHAPTER XIV

Reversionary Annuities

1. The temporary reversionary annuity

$$|_n a_{y|x} = |_n a_x - |_n a_{xy},$$

where (x)'s chance of receiving payments is confined to the first n years, must not be confused with the annuity to (x) after (y) for life which is to be entered upon only in the event of (y) dying within the first n years. To obtain the latter we must deduct from the reversionary annuity, $a_{y|x}$, the reversionary annuity after n years should both lives survive that period, $v^n {}_n p_{xy} a_{y+n|x+n}$. Its value will therefore be

$$(a_x - a_{xy}) - v^n {}_n p_{xy}(a_{x+n} - a_{x+n:y+n})$$

This again is different from the deferred reversionary annuity

$$\begin{aligned} {}_{n|}a_{y|x} &= {}_{n|}a_x - {}_{n|}a_{xy} \\ &= v^n {}_n p_x(1 - {}_n p_y)a_{x+n} + v^n {}_n p_{xy} a_{y+n|x+n} \end{aligned}$$

under which, as pointed out in *Text Book*, Article 5, it is not necessary that both lives should survive the period of deferment.

Another benefit to be distinguished from the foregoing is the annuity to commence on the death of (y) and continue during the subsequent lifetime of (x), but in any event to the end of n years certain from the present time. This is a reversionary annuity on $\overline{n|}$ after (y), $a_{y|\overline{n|}}$, together with a deferred reversionary annuity on (x) after (y), ${}_{n|}a_{y|x}$; then

$$a_{y|\overline{n|}} + {}_{n|}a_{y|x} = (a_{\overline{n|}} - a_{y\overline{n|}}) + ({}_{n|}a_x - {}_{n|}a_{xy})$$

Finally, this is different from the annuity-due to run for n years

certain after the death of (y), and for so much longer as (x) may live, the value of which is

$$\begin{aligned} A_y(1+a_{\overline{n-1}|})+a_{\overline{y(n)}|x} &= A_y(1+a_{\overline{n-1}|})+(a_x-a_{x:\overline{y(n)}}) \\ &= A_y(1+a_{\overline{n-1}|})+\frac{D_{x+n}}{D_x}(a_{x+n}-a_{x+n:y}) \end{aligned}$$

2. To find the annual premium for an endowment assurance to (x) payable at age $(x+n)$ or previous death, the premium to be doubled in the event of the death of (y) before (x) during the n years.

We have benefit side $= A_{x\overline{n}|}$

$$\begin{aligned} \text{Payment side} &= P(1+a_{x:\overline{n-1}|})+P\times {}_{|n-1}a_{y|x} \\ &= P(1+2a_{x:\overline{n-1}|}-a_{xy:\overline{n-1}|}) \end{aligned}$$

$$\text{Whence}\quad P = \frac{A_{x\overline{n}|}}{1+2a_{x:\overline{n-1}|}-a_{xy:\overline{n-1}|}}$$

The difficulty arises, however, that if (y) die early, say in the tth year, (x) may then obtain his benefit at a premium of $P_{x+t:\overline{n-t}|}$ which might be less than 2P as found above, and the office would not in this case obtain the premium on which it reckoned in making the contract.

3. To find the proportions in which the purchase price of a last-survivor annuity on (x) and (y) should be paid by them.

Each is entitled to half the annuity during the whole of his life, and to the remaining half for the period succeeding the death of the other life. That is, (x) is entitled to

$$\tfrac{1}{2}a_x+\tfrac{1}{2}a_{y|x} = a_x-\tfrac{1}{2}a_{xy}$$

and (y) is entitled to

$$\tfrac{1}{2}a_y+\tfrac{1}{2}a_{x|y} = a_y-\tfrac{1}{2}a_{xy}$$

$$\begin{aligned} \text{Now}\quad a_x-\tfrac{1}{2}a_{xy}+a_y-\tfrac{1}{2}a_{xy} &= a_x+a_y-a_{xy} \\ &= a_{\overline{xy}} \end{aligned}$$

which is the whole purchase price. Therefore (x) and (y) must pay $(a_x-\frac{1}{2}a_{xy})$ and $(a_y-\frac{1}{2}a_{xy})$ respectively.

4. To find the value of a last-survivor annuity on (x) and (y) which is to be reduced by half at the first death.

The whole annuity is payable during the joint lives, but half only during (x)'s life after (y), or (y)'s life after x. Therefore the value is

$$a_{xy} + \tfrac{1}{2}a_{y|x} + \tfrac{1}{2}a_{x|y} = \tfrac{1}{2}(a_x + a_y)$$

which is obviously correct.

5. The identity proved in *Text Book*, Article 18, may be very easily demonstrated by working from expression (1) to expression (10).

$$\begin{aligned}
a_{y|x} &= \Sigma v^t {}_tp_x(1 - {}_tp_y) \\
&= \Sigma v^t {}_tp_x(q_y + {}_{1|}q_y + {}_{2|}q_y + \cdot\ \cdot\ \cdot + {}_{t-1|}q_y) \\
&= vp_xq_y + v^2{}_2p_x(q_y + {}_{1|}q_y) + v^3{}_3p_x(q_y + {}_{1|}q_y + {}_{2|}q_y) + \text{ etc.} \\
&= \frac{vl_{x+1}d_y}{l_xl_y}(1 + vp_{x+1} + v^2{}_2p_{x+1} + \text{ etc.}) \\
&\quad + \frac{v^2l_{x+2}d_{y+1}}{l_xl_y}(1 + vp_{x+2} + v^2{}_2p_{x+2} + \text{ etc.}) \\
&\quad + \frac{v^3l_{x+3}d_{y+2}}{l_xl_y}(1 + vp_{x+3} + v^2{}_2p_{x+3} + \text{ etc.}) \\
&\quad + \text{ etc.} \\
&= \Sigma v^t \frac{l_{x+t}d_{y+t-1}}{l_xl_y}(1 + a_{x+t})
\end{aligned}$$

6. In *Text Book*, Article 19, the effect of a change in the mortality to which (x) is subject is pointed out. But similarly the mortality experience of the two lives may be assumed throughout to be that shown by two separate tables. Thus in the annuity $a_{x|y}$, the assumption may be made that the $O^{[NM]}$ Table shows the mortality of the class to which (x) belongs and $O^{[af]}$ that of (y). Then

$$a_{[x]|[y]} = a_{[y]} - a_{[x][y]}$$

where $a_{[y]}$ is taken from the $O^{[af]}$ Table, and $a_{[x][y]}$ is calculated with $O^{[NM]}$ mortality for (x) and $O^{[af]}$ mortality for (y). To get the annual premium we shall divide by $\text{a}_{[x][y]}$ found as above.

It is to be noted that (x) must be medically examined, but (y) need not, as the latter's early death is to the advantage of the office.

7. Commutation columns for the calculation of reversionary annuities could be formed if necessary.

For $$a_{y|x} = \frac{vl_{x+1}d_y(1+a_{x+1})+v^2l_{x+2}d_{y+1}(1+a_{x+2})+\cdots}{l_x l_y}$$

Following Davies's principle,

$$= \frac{v^{x+1}l_{x+1}d_y(1+a_{x+1})+v^{x+2}l_{x+2}d_{y+1}(1+a_{x+2})+\cdots}{v^x l_x l_y}$$

$$= \frac{{}^{(ra)}C_{\overset{1}{xy}} + {}^{(ra)}C_{\overline{x+1}:\overset{1}{\overline{y+1}}} + \cdots}{D_{xy}} \quad \text{if } x > y$$

$$= \frac{{}^{(ra)}M_{\overset{1}{xy}}}{D_{xy}}$$

or $$= \frac{{}^{(ra)}C_{\overset{1}{xy}} + {}^{(ra)}C_{\overline{x+1}:\overset{1}{\overline{y+1}}} + \cdots}{(1+i)^{y-x}D_{xy}} \quad \text{if } x < y$$

$$= \frac{{}^{(ra)}M_{\overset{1}{xy}}}{(1+i)^{y-x}D_{xy}}$$

D_{xy} is the ordinary joint-life commutation value, and the forms of ${}^{(ra)}C_{\overset{1}{xy}}$ and ${}^{(ra)}M_{\overset{1}{xy}}$ are clear from the above.

If the annuity is to be entered upon at the death of (y) it takes the form,

$$\hat{a}_{y|x} = \frac{v^{\frac{1}{2}}l_{x+\frac{1}{2}}d_y a_{x+\frac{1}{2}} + v^{\frac{3}{2}}l_{x+\frac{3}{2}}d_{y+1}a_{x+\frac{3}{2}}+\cdots}{l_x l_y}$$

$$= \frac{v^{x+\frac{1}{2}}l_{x+\frac{1}{2}}d_y a_{x+\frac{1}{2}} + v^{x+\frac{3}{2}}l_{x+\frac{3}{2}}d_{y+1}a_{x+\frac{3}{2}}+\cdots}{v^x l_x l_y}$$

$$= \frac{{}^{(ra)}\hat{C}_{\overset{1}{xy}} + {}^{(ra)}\hat{C}_{\overline{x+1}:\overset{1}{\overline{y+1}}} + \cdots}{D_{xy}} \quad \text{if } x > y$$

$$= \frac{{}^{(ra)}\hat{M}_{\overset{1}{xy}}}{D_{xy}}$$

or $$= \frac{{}^{(ra)}\hat{C}_{\overset{1}{xy}} + {}^{(ra)}\hat{C}_{\overline{x+1}:\overset{1}{\overline{y+1}}} + \cdots}{(1+i)^{y-x}D_{xy}} \quad \text{if } x < y$$

$$= \frac{{}^{(ra)}\hat{M}_{\overset{1}{xy}}}{(1+i)^{y-x}D_{xy}}$$

Here, again, the method of forming ${}^{(ra)}\hat{\mathrm{C}}_{\overset{1}{xy}}$ and ${}^{(ra)}\hat{\mathrm{M}}_{\overset{1}{xy}}$ is sufficiently clear.

If the annuity is to be entered on at the death of (y) with a proportionate payment to the date of death of (x), we have

$$\hat{a}_{y|x} = \frac{v^{\frac{1}{2}}l_{x+\frac{1}{2}}d_y\left(a_{x+\frac{1}{2}}+\frac{1}{2}\bar{\mathrm{A}}_{x+\frac{1}{2}}\right)+v^{\frac{3}{2}}l_{x+\frac{3}{2}}d_{y+1}\left(a_{x+\frac{3}{2}}+\frac{1}{2}\bar{\mathrm{A}}_{x+\frac{3}{2}}\right)+\cdots}{l_x l_y}.$$

The denominator will take the form D_{xy} or $(1+i)^{y-x}\mathrm{D}_{xy}$ according as $x >$ or $< y$; and the numerator in either case will take the form

$${}^{(ra)}\mathring{\hat{\mathrm{C}}}_{\overset{1}{xy}}+{}^{(ra)}\mathring{\hat{\mathrm{C}}}_{\overline{x+1}:\overset{1}{\overline{y+1}}}+\cdots={}^{(ra)}\mathring{\hat{\mathrm{M}}}_{\overset{1}{xy}}$$

where ${}^{(ra)}\mathring{\hat{\mathrm{C}}}_{\overset{1}{xy}} = v^{x+\frac{1}{2}}l_{x+\frac{1}{2}}d_y\left(a_{x+\frac{1}{2}}+\frac{1}{2}\bar{\mathrm{A}}_{x+\frac{1}{2}}\right)$

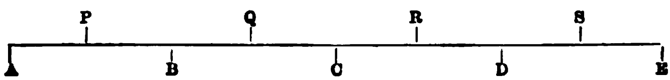
Tables must therefore be formed of this function for every value of x with every value of y, and the summation of these values from x upwards where $x-y$ remains constant will give ${}^{(ra)}\mathring{\hat{\mathrm{M}}}_{\overset{1}{xy}}$ as above.

8. On consideration of the argument in *Text Book*, Articles 21, 23, and 26, we have clearly on the analogy of formula (10)

$$\begin{aligned} a^{(m)}_{y|x} &= \Sigma v^t {}_tp_x({}_{t-1}p_y - {}_tp_y)\mathrm{a}^{(m)}_{x+t} \\ &= \Sigma v^t {}_tp_x({}_{t-1}p_y - {}_tp_y)\left(\mathrm{a}_{x+t}-\frac{m-1}{2m}\right) \\ &= a_{y|x}-\frac{m-1}{2m}\left(\frac{a_{x:y-1}}{p_{y-1}}-a_{xy}\right) \end{aligned}$$

which is *Text Book* formula (14).

9. The method of finding the value of $\hat{a}^{(m)}_{y|x}$ given in *Text Book*, Article 30, may be made more clear by a simple graphic illustration.



etc.

Let A, B, C, etc., represent the ends of $m$thly intervals as from the date of effecting the contract; and let P, Q, R, etc., be at the middle of these intervals.

Now on the assumption of the deaths being equally distributed throughout each interval, if (y) die, say, in the interval AB, his death will on the average occur at P. Therefore in the case of the formula,

$$a^{(m)}_{y|x} = a^{(m)}_x - a^{(m)}_{xy}$$

the first payment of $\frac{1}{m}$ will be made at B, and the succeeding payments at C, D, etc.

But in the case of the benefit represented by $\mathring{a}^{(m)}_{y|x}$ it is desired that the first payment of $\frac{1}{m}$ should be made at Q, and the succeeding payments at R, S, etc. If therefore the formula $a^{(m)}_{y|x}$ is to be employed, it is clear that not $\frac{1}{m}$ but $v^{\frac{1}{2m}}\frac{1}{m}$ should be paid at B, C, D, etc., for this will amount with interest to $\frac{1}{m}$ at the correct dates of payment Q, R, S, etc.

If the transaction be effected thus, (x) will receive a payment of $v^{\frac{1}{2m}}\frac{1}{m}$, to which, on the conditions of the benefit $\mathring{a}^{(m)}_{y|x}$, he is not entitled, in the event of his dying between B and Q or between C and R, etc. Making a deduction for this over-payment and keeping to our previous assumption as to equal distribution of deaths, we have the value of the correction as shown in *Text Book*, Article 30, $= \frac{1}{2m}\,A^2_{xy}$ approximately.

A further correction is necessary to make the annuity complete, and the value of this is also approximately $\frac{1}{2m}\,A^2_{xy}$ as shown.

These two corrections are equal in value but opposite in sign, and we therefore have finally,

$$\mathring{a}^{(m)}_{y|x} = v^{\frac{1}{2m}}\left(a^{(m)}_x - a^{(m)}_{xy}\right)$$

10. To find $_{|n}\mathring{a}^{(m)}_{y|x}$.

As before, let payments of $v^{\frac{1}{2m}}\frac{1}{m}$ be made under the annuity

$|_n a_x^{(m)} - |_n a_{xy}^{(m)}$ at B, C, D, etc., which will accumulate to $\frac{1}{m}$ at Q, R, S, etc.: the value of this annuity being,

$$v^{\frac{1}{2m}}(|_n a_x^{(m)} - |_n a_{xy}^{(m)}).$$

Now let E be the end of the n years. At E a payment of $v^{\frac{1}{2m}}\frac{1}{m}$ is due, but is not to be made, because the annuity is to cease at the end of n years and the payment of $\frac{1}{m}$ to which this would accumulate by, say, T falls outside that period, and accordingly is not payable. A deduction similar to that for the whole-of-life reversionary annuity must also be made. We then have,

$$|_n \hat{a}_{y|x}^{(m)} = v^{\frac{1}{2m}}(|_n a_x^{(m)} - |_n a_{xy}^{(m)}) - \frac{1}{m} v^{\frac{1}{2m}} v^n {}_n p_x (1 - {}_n p_y) - \frac{1}{2m} \times |_n A_{xy}^2$$

Besides an addition to make the annuity complete in the same manner as for the whole reversionary annuity we must add a proportion for the period between S and E. Therefore

$$\begin{aligned} |_n \hat{\hat{a}}_{y|x}^{(m)} &= v^{\frac{1}{2m}}(|_n a_x^{(m)} - |_n a_{xy}^{(m)}) - \left\{\frac{1}{m} v^{\frac{1}{2m}} v^n {}_n p_x (1 - {}_n p_y) + \frac{1}{2m} \times |_n A_{xy}^2\right\} \\ &\quad + \left\{\frac{1}{2m} \times |_n A_{xy}^2 + \frac{1}{2m} v^n {}_n p_x (1 - {}_n p_y)\right\} \\ &= v^{\frac{1}{2m}}(|_n a_x^{(m)} - |_n a_{xy}^{(m)}) - \left(\frac{1}{m} v^{\frac{1}{2m}} - \frac{1}{2m}\right) v^n {}_n p_x (1 - {}_n p_y) \end{aligned}$$

11. To find ${}_n|\hat{\hat{a}}_{y|x}^{(m)}$.

$$\begin{aligned} \text{We have } {}_n|a_{y|x}^{(m)} &= {}_n|a_x^{(m)} - {}_n|a_{xy}^{(m)} \\ &= v^n {}_n p_x a_{x+n}^{(m)} - v^n {}_n p_{xy} a_{x+n:y+n}^{(m)} \\ &= v^n {}_n p_x (1 - {}_n p_y) a_{x+n}^{(m)} + v^n {}_n p_{xy} a_{y+n|x+n}^{(m)} \end{aligned}$$

With regard to the first portion of this formula, if (x) survive and (y) die within the n years, the first payment of the annuity $a_{x+n}^{(m)}$ will be made at the end of $\frac{1}{m}$ of a year after n years, and therefore

$a^{(m)}_{x+n} = \mathring{a}^{(m)}_{x+n}$. Also in regard to the second part, $\hat{\mathring{a}}^{(m)}_{y+n|x+n}$ is of the same nature as $\hat{\mathring{a}}^{(m)}_{y|x}$, which has been already discussed.

Therefore,

$$\begin{aligned}
{}_{n|}\hat{\mathring{a}}^{(m)}_{y|x} &= v^n{}_np_x(1-{}_np_y)\mathring{a}^{(m)}_{x+n} + v^n{}_np_{xy}\hat{\mathring{a}}^{(m)}_{y+n|x+n} \\
&= v^n{}_np_x(1-{}_np_y)\left(a^{(m)}_{x+n} + \frac{1}{2m}\mathrm{A}_{x+n}\right) \\
&\quad + v^n{}_np_{xy}v^{\frac{1}{2m}}(a^{(m)}_{x+n} - a^{(m)}_{x+n:y+n})
\end{aligned}$$

In the above formula $\frac{1}{2m}\bar{\mathrm{A}}_{x+n}$ would be more exact than $\frac{1}{2m}\mathrm{A}_{x+n}$, but all through this chapter correction for payment of claims at moment of death is ignored.

12. By *Text Book* formula (22) the annual premium for a reversionary annuity is found to be

$$\mathrm{P}a_{y|x} = \frac{a_x - a_{xy}}{1 + a_{xy}}$$

Now the value of a reversionary annuity occasionally decreases as the lives grow older, and therefore the annual premium for an annuity to (x) after (y) may be greater than that for a similar annuity to $(x+1)$ after $(y+1)$. This is a state of matters the reverse of what is usually found in assurance contracts, with the consequence that a level premium to be charged throughout may be too small at first and afterwards too great; and were the assured to realise this they might drop the policy and get the same benefit for a lighter premium after a few years. The first year's risk is $vp_xq_y(1+a_{x+1}) = q_ya_x$, and $\mathrm{P}a_{y|x}$ must never be less than this.

13. To find the reversionary annuity to two children aged 10 and 15 respectively, which is to commence on the death of their mother aged 50 and to continue till both children have attained majority.

This is a temporary annuity to (10) after (50) for 11 years, together with a temporary annuity to (15) after (50) for 6 years,

less a temporary annuity during the joint lives (10) and (15) after the death of (50) for 6 years. That is,

$$|_{11}a_{50}|_{10} + |_{6}a_{50}|_{15} - |_{6}a_{50}|_{10:15}$$
$$= |_{11}a_{10} - |_{11}a_{50:10} + |_{6}a_{15} - |_{6}a_{50:15} - |_{6}a_{10:15} + |_{6}a_{50:10:15}$$
$$= a_{\overline{(10:\overline{11|})(15:\overline{6|})}} - a_{50:\overline{(10:\overline{11|})(15:\overline{6|})}}$$

14. To find $a_{\overline{xy}|\overline{ab}}$

(a)
$$\begin{aligned} a_{\overline{xy}|\overline{ab}} &= a_{\overline{ab}} - a_{\overline{ab}:\overline{xy}} \\ &= a_{\overline{abxy}} - a_{\overline{xy}} \\ &= (a_a + a_b + a_x + a_y) - (a_{ab} + a_{ax} + a_{ay} + a_{bx} + a_{by} + a_{xy}) \\ &\qquad + (a_{abx} + a_{aby} + a_{axy} + a_{bxy}) - a_{abxy} \\ &\qquad - (a_x + a_y) + a_{xy} \\ &= (a_a + a_b) - (a_{ab} + a_{ax} + a_{ay} + a_{bx} + a_{by}) \\ &\qquad + (a_{abx} + a_{aby} + a_{axy} + a_{bxy}) - a_{abxy} \end{aligned}$$

(b)
$$\begin{aligned} a_{\overline{xy}|\overline{ab}} &= a_{\overline{xy}|a} + a_{\overline{xy}|b} - a_{\overline{xy}|ab} \\ &= (a_{x|a} + a_{y|a} - a_{xy|a}) + (a_{x|b} + a_{y|b} - a_{xy|b}) \\ &\qquad - (a_{x|ab} + a_{y|ab} - a_{xy|ab}) \\ &= (a_a - a_{ax} + a_a - a_{ay} - a_a + a_{axy}) \\ &\qquad + (a_b - a_{bx} + a_b - a_{by} - a_b + a_{bxy}) \\ &\qquad - (a_{ab} - a_{abx} + a_{ab} - a_{aby} - a_{ab} + a_{abxy}) \\ &= (a_a + a_b) - (a_{ax} + a_{ay} + a_{bx} + a_{by} + a_{ab}) \\ &\qquad + (a_{axy} + a_{bxy} + a_{abx} + a_{aby}) - a_{abxy} \end{aligned}$$

15. The problem of Endowment Assurance Instalment Policies discussed at page 148 is sometimes further complicated by the introduction of a beneficiary. If the life assured die before the date of maturity, the beneficiary is to receive an annuity for life with the guarantee of n payments certain; but if, on the other hand, the life assured survive the endowment period, then the annuity guaranteed at that date is for n years certain, and continued beyond the n years to the last survivor of the life assured

and the beneficiary. This extra benefit for the first $(m+n-1)$ years is the value of an annuity of $\frac{1}{n}$ to (y), which however is not payable so long as (x) is alive, nor for n years after, or, in symbols,

$$\frac{1}{n}\left(|_{m+n-1}a_y - |_{m+n-1}a_{y:\overline{x(\overline{n|})}}\right)$$

And for the period after $(m+n-1)$ years, the benefit is a reversionary annuity to (y) after (x) deferred $(m+n-1)$ years; in symbols,

$$\frac{1}{n} \times {}_{m+n-1|}a_{x|y}$$

The whole extra benefit is therefore

$$\frac{1}{n}\left(|_{m+n-1}a_y - |_{m+n-1}a_{y:\overline{x(\overline{n|})}} + {}_{m+n-1|}a_{x|y}\right)$$

$$= \frac{1}{n}\left(a_{y:\overline{m+n-1|}} - a_{y\overline{n|}} - \frac{D_{y+n}}{D_y}a_{y+n:x:\overline{m-1|}} + {}_{m+n-1|}a_{x|y}\right)$$

(See value of $|_m a_{x:\overline{y(\overline{n|})}}$ deduced on page 131.)

To get the extra annual premium, we must divide this function by $a_{xy\overline{m|}}$ in order to guard against an option as already discussed. This portion of the premium will cease to be payable on the death of (y) before (x).

EXAMPLES

1. An annuity of £100 per annum, payable until the death of the last survivor of three lives, A, B, and C, aged respectively 20, 30, and 40, is to be divided equally between A and B during their joint lives, afterwards between the survivor and C, if living, and ultimately is payable to the last survivor. Find the value of A's interest.

Given	$a_{20} = 20{\cdot}2246$		$a_{20:30} = 16{\cdot}1739$
	$a_{30} = 18{\cdot}4156$		$a_{30:40} = 13{\cdot}9872$
	$a_{40} = 16{\cdot}1026$		$a_{20:40} = 14{\cdot}5274$
	$a_{47} = 14{\cdot}1097$		$a_{20:47} = 12{\cdot}9502$
	$a_{48} = 13{\cdot}8064$		$a_{20:48} = 12{\cdot}7015$

A's interest $= \frac{1}{2}a_{20:30} + \frac{1}{2}a_{30|20:40} + a_{\overline{30:40}|20}$

$= \frac{1}{2}a_{20:30} + (\frac{1}{2}a_{20:40} - \frac{1}{2}a_{20:30:40}) + (a_{20} - a_{20:30} - a_{20:40} + a_{20:30:40})$

$= a_{20} - \frac{1}{2}a_{20:30} - \frac{1}{2}a_{20:40} + \frac{1}{2}a_{20:30:40}$

To find the value of $a_{20:30:40}$ we have

$$a_{30:40} = 13{\cdot}9872 = a_{47+\kappa}$$

whence by Milne's modification of Simpson's rule, the age of the oldest life being less than 45, we assume $w = 48$, and

$$a_{20:30:40} = a_{20:48\cdot} = 12{\cdot}7015$$

Therefore $\quad a_{20} - \frac{1}{2}a_{20:30} - \frac{1}{2}a_{20:40} + \frac{1}{2}a_{20:30:40}$

$$= 20{\cdot}2246 - \tfrac{1}{2}(16{\cdot}1739 + 14{\cdot}5274 - 12{\cdot}7015)$$

$$= 20{\cdot}2246 - 8{\cdot}9999$$

$$= 11{\cdot}2247$$

Multiplying by 100 we have A's interest = £1122, 9s. 5d. nearly.

2. Find the value of a reversionary annuity payable so long as three at least of the four lives (w), (x), (y), and (z) live after the death of (f).

$$a_{f|\overline{wxyz}}^{\;3} = \Sigma v^t(1 - {}_tp_f)\,{}_tp_{\overline{wxyz}}^{\;3}$$

$$= \Sigma v^t(1 - {}_tp_f)({}_tp_{wxy} + {}_tp_{wxz} + {}_tp_{wyz} + {}_tp_{xyz} - 3\,{}_tp_{wxyz})$$

$$= a_{f|wxy} + a_{f|wxz} + a_{f|wyz} + a_{f|xyz} - 3a_{f|wxyz}$$

which may be reduced to terms of joint-life annuities on three, four, and five lives.

The general formula is

$$a_{f|\overline{wxyz\ldots(m)}}^{\;r} = \Sigma v^t(1 - {}_tp_f)\,{}_tp_{\overline{wxyz\ldots(m)}}^{\;r}$$

3. What is the value of an annuity during the lives of (x), (y), and (z), £100 a year to be paid so long as they are all alive, £80

a year after the first death, and £60 a year after the second death?

The value of the annuity is

$$100\,a_{xyz} + 80(a_{z}|_{yz} + a_{y}|_{xz} + a_{z}|_{xy}) + 60(a_{\overline{xy}}|_{z} + a_{\overline{xz}}|_{y} + a_{\overline{yz}}|_{x})$$

4. An annuity-certain of £a for the term of n years is to be enjoyed by P and his heirs during the joint existence of two lives aged x and y; and if that joint existence fail before the expiration of m years, the annuity is to go to Q and his heirs for the remainder of the term of n years. Determine the value of Q's interest.

Q gets a reversionary annuity for the remainder of the term of n years after the failure of the joint lives x and y, but only in the event of that failure taking place within m years.

Therefore from the reversionary annuity

$$a_{\overline{n}|} - a_{xy:\overline{n}|}$$

we must deduct the portion thereof after m years should both lives survive, or

$$v^{m}{}_{m}p_{xy}(a_{\overline{n-m}|} - a_{x+m:y+m:\overline{n-m}|})$$

Q's interest is therefore

$$£a\{(a_{\overline{n}|} - a_{xy:\overline{n}|}) - v^{m}{}_{m}p_{xy}(a_{\overline{n-m}|} - a_{x+m:y+m:\overline{n-m}|})\}$$

5. Determine the respective interests of (x), (y), and (z) in an annuity payable so long as at least two of them are alive; and to be divided between (x) and (y) equally during their joint lives, and after the death of either in like proportions between (z) and the survivor during what may remain of their joint lives.

The whole annuity is

$$a_{\overline{xyz}}^{\;2} = a_{xy} + a_{xz} + a_{yz} - 2a_{xyz}$$

The share falling to (x) is half of the joint-life annuity a_{xy} together with half of the annuity to the joint lives (x) and (z) after the death of (y), or

$$\tfrac{1}{2}a_{xy} + \tfrac{1}{2}a_{y|xz} = \tfrac{1}{2}a_{xy} + \tfrac{1}{2}a_{xz} - \tfrac{1}{2}a_{xyz}$$

The share falling to (y) is similar to (x)'s. The relative positions of the two have merely to be changed. Therefore (y)'s share is

$$\tfrac{1}{2}a_{xy} + \tfrac{1}{2}a_{z|yz} = \tfrac{1}{2}a_{xy} + \tfrac{1}{2}a_{yz} - \tfrac{1}{2}a_{xyz}$$

(z) gets half of the annuity so long as he survives jointly with (x) after (y)'s death, or with (y) after (x)'s death, or

$$\begin{aligned} \tfrac{1}{2}a_{y|xz} + \tfrac{1}{2}a_{x|yz} &= \tfrac{1}{2}a_{xz} - \tfrac{1}{2}a_{xyz} + \tfrac{1}{2}a_{yz} - \tfrac{1}{2}a_{xyz} \\ &= \tfrac{1}{2}a_{xz} + \tfrac{1}{2}a_{yz} - a_{xyz} \end{aligned}$$

The three shares together make up the whole annuity

$$\begin{aligned} (\tfrac{1}{2}a_{xy} + \tfrac{1}{2}a_{xz} - \tfrac{1}{2}a_{xyz}) + (\tfrac{1}{2}a_{xy} + \tfrac{1}{2}a_{yz} - \tfrac{1}{2}a_{xyz}) + (\tfrac{1}{2}a_{xz} + \tfrac{1}{2}a_{yz} - a_{xyz}) \\ = a_{xy} + a_{xz} + a_{yz} - 2a_{xyz} \quad \text{as above.} \end{aligned}$$

6. An annuity of 1 is granted for 15 years to the last survivor of (x), (y), and (z). So long as they are all alive during the first 5 years, the annuity is to be paid to (x) alone; and, so long as they are all alive during the second 5 years, to (y) alone. At the expiry of 10 years, or at the first death, the annuity is to be divided equally between the survivors, and is to go wholly to the last survivor. Express the value of (y)'s interest in the annuity.

The value of (y)'s interest is

$${}_{5|5}a_{xyz} + \tfrac{1}{3} \times {}_{10|5}a_{xyz} + \tfrac{1}{2}\left({}_{|15}a_{x|yz} + {}_{|15}a_{z|xy}\right) + {}_{|15}a_{\overline{xz}|y}$$

7. An annuity of 1 to the last survivor of three lives (x), (y), and (z) is to be divided equally between (x) and (y) during their joint lives; if (x) dies first, (y) and (z) are to enjoy it equally during their joint lives and the survivor of them is to have the whole; but if (y) dies first, (x) is to enjoy the whole during his life, and after his decease the whole annuity goes to (z). Find the value of their respective shares.

The value of (x)'s share

$$\begin{aligned} &= \tfrac{1}{2}a_{xy} + a_{y|x} \\ &= a_x - \tfrac{1}{2}a_{xy} \end{aligned}$$

The value of (y)'s share

$$\begin{aligned}
&= \tfrac{1}{2}a_{xy} + \tfrac{1}{2}a_{z|yz} + a_{\overline{xz}|y} \\
&= \tfrac{1}{2}(a_{xy} + a_{yz} - a_{xyz} + 2a_y - 2a_{xy} - 2a_{yz} + 2a_{xyz}) \\
&= a_y - \tfrac{1}{2}(a_{xy} + a_{yz} - a_{xyz}) \\
&= a_y - \tfrac{1}{2}a_{y:\overline{xz}}
\end{aligned}$$

And the value of (z)'s share

$$\begin{aligned}
&= \tfrac{1}{2}a_{x|yz} + a_{\overline{xy}|z} \\
&= \tfrac{1}{2}(a_{yz} - a_{xyz} + 2a_z - 2a_{xz} - 2a_{yz} + 2a_{xyz}) \\
&= (a_z - a_{xz}) - \tfrac{1}{2}(a_{yz} - a_{xyz})
\end{aligned}$$

The sum of the three shares makes up $a_{\overline{xyz}}$, the whole annuity.

8. Find a formula for the value of a reversionary annuity payable for the remainder of the life of B, after the death of A, the annuity being reducible by one-half should such death not occur for 7 years, and by two-thirds should it not occur for 10 years from the present time.

If A be aged x, and B be aged y, we have the value of the annuity

$$a_{x|y} - \tfrac{1}{2}v^7\,{}_7p_{xy}\,a_{x+7|y+7} - \tfrac{1}{6}v^{10}\,{}_{10}p_{xy}\,a_{x+10|y+10}$$

9. An annuity of £k is to be paid to (z) so long as he and (x) and (y) are all alive. At the first death of the three lives the annuity is to be shared equally by the survivors; and, at the second death, it is to be continued for n years certain to the last survivor of the lives or to his heirs. Express the share of (z) in the annuity.

$$\begin{aligned}
&£k\{a_{xyz} + \tfrac{1}{2}(a_{y|xz} + a_{x|yz}) + \mathrm{A}_{\overline{xy}:\overset{}{\underset{2}{z}}}(1 + a_{\overline{n-1|}})\} \\
= \; &£k\{\tfrac{1}{2}(a_{xz} + a_{yz}) + (\mathrm{A}^{1}_{xz} - \mathrm{A}^{1}_{xyz} + \mathrm{A}^{1}_{yz} - \mathrm{A}_{xyz}^{\;\;1})(1 + a_{\overline{n-1|}})\}
\end{aligned}$$

10. There are two formulas for $a_{y|x}$, viz. (a) $\Sigma v^t\,{}_tp_x(1 - {}_tp_y)$ and (b) $\Sigma v^t\,{}_{t-1|}q_y \times {}_tp_x(1 + a_{x+t})$. Give the corresponding formulas for $a_{\overline{yz}|x}$, and prove their identity.

$$(a)\quad a_{\overline{yz}|x} = \Sigma v^t\,{}_tp_x(1 - {}_tp_y)(1 - {}_tp_z)$$

$$(b)\quad a_{\overline{yz}|x} = \Sigma v^t\,{}_{t-1|}q_{\overline{yz}} \times {}_tp_x(1 + a_{x+t})$$

$$\begin{aligned}
\Sigma v^t{}_tp_x(1-{}_tp_y)(1-{}_tp_z) &= \Sigma v^t{}_tp_x \times {}_{|t}q_{\overline{yz}} \\
&= \Sigma v^t{}_tp_x(q_{\overline{yz}} + {}_{1|}q_{\overline{yz}} + \cdots + {}_{t-1|}q_{\overline{yz}}) \\
&= vp_xq_{\overline{yz}} + v^2{}_2p_x(q_{\overline{yz}} + {}_{1|}q_{\overline{yz}}) + v^3{}_3p_x(q_{\overline{yz}} + {}_{1|}q_{\overline{yz}} + {}_{2|}q_{\overline{yz}}) + \cdots \\
&= vp_xq_{\overline{yz}}(1 + vp_{x+1} + v^2{}_2p_{x+1} + \cdots) \\
&\qquad + v^2{}_2p_x \times {}_{1|}q_{\overline{yz}}(1 + vp_{x+2} + \cdots) \\
&\qquad + \cdots \\
&= \Sigma v^t{}_tp_x \times {}_{t-1|}q_{\overline{yz}}(1 + a_{x+t})
\end{aligned}$$

11. A father aged 50 wishes to secure to his two children aged 8 and 10 respectively, an annuity of £n, to commence at his death and to continue until the younger child, or the elder if he be the survivor, attains the age of 21. Find a formula for the value of the annuity. Would it be safe to grant such an annuity to be secured by annual premiums?

The value of the annuity is

$$£n({}_{|13}a_{50|8} + {}_{|11}a_{50|10} - {}_{|11}a_{50|8:10})$$

One difficulty in connection with accepting payment by annual premiums is in determining the period during which the premiums should be made payable.

If they are made payable throughout the whole status, the risk is run of the contract being dropped, and of a new one being effected at a cheaper premium in the event of the early failure of one of the children's lives. In this case to obtain the annual premium we should divide the single premium by

$$(1 + a_{50:8:\overline{12|}} + a_{50:10:\overline{10|}} - a_{50:8:10:\overline{10|}})$$

If, on the other hand, the premium is accepted only during the joint status, there is a risk of either (8) or (10) being a bad life, the premium payable being considerably underestimated as a consequence. In this case we should divide by $(1 + a_{50:8:10:\overline{10|}})$.

It is possible, too, that even if all survive, the annual premium for a similar benefit for the remainder of the term may diminish, in spite of the increase in the ages.

A contract by annual premiums should therefore, if possible, be avoided.

12. Find, according to the *Text Book* Table, the value, at 3 per cent. interest, of a contingent annuity for the remainder of 30 years certain from the present time, the annuity to commence

on the failure of the joint existence of two lives both now aged 30, but only in the event of such failure taking place after the expiration of 5 years and before the completion of 10 years.

The formula to be used is

$$v^5{}_5p_{30:30} \times a_{35:35|\overline{25}|} - v^{10}{}_{10}p_{30:30} \times a_{40:40|\overline{20}|}$$

$$= \frac{D_{35:35}}{D_{30:30}}\left(a_{\overline{25}|} - a_{35:35} + \frac{D_{60:60}}{D_{35:35}}a_{60:60}\right)$$

$$- \frac{D_{40:40}}{D_{30:30}}\left(a_{\overline{20}|} - a_{40:40} + \frac{D_{60:60}}{D_{40:40}}a_{60:60}\right)$$

$$= \frac{D_{35:35}}{D_{30:30}}(a_{\overline{25}|} - a_{35:35}) - \frac{D_{40:40}}{D_{30:30}}(a_{\overline{20}|} - a_{40:40})$$

$$= 1{\cdot}807 - {\cdot}726$$

$$= 1{\cdot}081$$

$\log D_{35:35}$ = 9·42108		$\log D_{40:40}$ = 9·31707	
$\log D_{30:30}$ = 9·52032		$\log D_{30:30}$ = 9·52032	
$\bar{1}{\cdot}90076$		$\bar{1}{\cdot}79675$	
$a_{\overline{25}|}$ = 17·413		$a_{\overline{20}|}$ = 14·877	
$a_{35:35}$ = 15·142		$a_{40:40}$ = 13·718	
2·271		1·159	
log 2·271 = ·35622		log 1·159 = ·06408	
$\bar{1}{\cdot}90076$		$\bar{1}{\cdot}79675$	
·25698		$\bar{1}{\cdot}86083$	
= log 1·807		= log ·726	

13. Required the annual premium for an annuity payable to the last survivor of (x) and (y), to commence at the end of n years if both be living, or at the first death if that occur within n years.

The benefit side

$$= v^n{}_np_{xy}a_{\overline{x+n:y+n}} + a_{y|x} - v^n{}_np_{xy}a_{y+n|x+n}$$

$$+ a_{x|y} - v^n{}_np_{xy}a_{x+n|y+n}$$

$$= v^n{}_np_{xy}(a_{x+n} + a_{y+n} - a_{x+n:y+n} - a_{x+n} + a_{x+n:y+n}$$

$$- a_{y+n} + a_{x+n:y+n}) + a_x - a_{xy} + a_y - a_{xy}$$

$$= a_x + a_y - 2a_{xy} + v^n{}_np_{xy}a_{x+n:y+n}$$

This might be written $a_{\overline{xy}} - {}_{|n}a_{xy}$, and in this form it means an annuity to the last survivor, not payable, however, so long as they both survive within the first n years; which is the required annuity expressed in other terms. Our reasoning is thus proved correct.

$$\text{The payment side} = \text{P}\mathrm{a}_{xy\overline{n}|}$$

$$\text{And P} = \frac{a_x + a_y - 2a_{xy} + v^n {}_np_{xy} a_{x+n:y+n}}{\mathrm{a}_{xy\overline{n}|}}$$

14. Ascertain the annual premium for a reversionary annuity which (x) desires to provide for his wife (y) and child (z) after his death. The annuity is to be £100 so long as (y) survives, but to be reduced to £50 at (y)'s death.

The benefit side consists of two parts: (a) an annuity to (y) after (x)'s death; and (b) an annuity to (z) after the death of the survivor of (x) and (y); that is

$$100a_{x|y} + 50a_{\overline{xy}|z}$$

A difficulty arises in determining the status for payment of the premium. If the annuity were not subject to reduction on (y)'s death the premium might be made payable during the joint life of (x) and the survivor of (y) and (z). But in this case, should (y) die early, the reversionary annuity to (z) might be obtained at a smaller premium than that so found. Again, if the premium were made payable during the joint existence of (x) and (y), and (y) were to die early (she not being subject to medical examination), the contract is practically a reversionary annuity to (z) after (x) at an insufficient premium. The best plan is, if possible, to have separate contracts for the two portions of the benefit, and have the premium for the former payable during the joint lives of (x) and (y), and for the latter during the joint lives of (x) and (z).

15. Under a deed of separation, A covenants to pay an annuity of £K per annum to his wife B so long as she lives, and the terms of the deed make his estate liable for the annuity after his death. He wishes to free his estate from this liability so long as his daughter C survives him, and he applies to an insurance office for a quotation for the annual premium for such a contingent annuity. Find the net annual premium.

The company will have to pay the annuity so long as C

survives jointly with B after the death of A, and the premium will cease on the first death of A, B, and C.

Therefore if the ages of A, B, and C are x, y, and z respectively, the value of the payments is

$$\mathrm{K} \times a_{x|yz} = \mathrm{K}(a_{yz} - a_{xyz})$$

and of the premiums $\quad \mathrm{P}(1 + a_{xyz})$

whence equating and solving $\mathrm{P} = \dfrac{\mathrm{K}(a_{yz} - a_{xyz})}{1 + a_{xyz}}$

16. Determine the annual premium for an annuity of s to continue during the lifetime of B aged y, after the death of A, aged x; with the proviso that should A survive the age of $x+n$ a sum of t is to be at once paid to B, if then alive, instead of the annuity.

The value of the benefit is

$$s\{(a_y - a_{yx}) - v^n {}_np_{xy}(a_{y+n} - a_{y+n:x+n})\} + tv^n {}_np_{xy}$$

The value of the premium which will run during the joint existence for n years is

$$\mathrm{P}(1 + a_{xy:\overline{n-1}|})$$

$$\text{Hence} \quad \mathrm{P} = \frac{s\left\{(a_y - a_{yx}) - \dfrac{\mathrm{D}_{x+n:y+n}}{\mathrm{D}_{xy}}(a_{y+n} - a_{y+n:x+n})\right\} + t\,\dfrac{\mathrm{D}_{x+n:y+n}}{\mathrm{D}_{xy}}}{1 + a_{xy:\overline{n-1}|}}$$

17. Give the formula, reduced to its simplest form, for the annual premium for an annuity of 1 to a female aged y to be entered on at the death of her husband aged x if that occur within the next 20 years; but to be entered on at the end of 20 years if (y) be then alive, whether her husband survive that time or not. The annuity is to be payable by half-yearly instalments, and with a proportion to date of death of the annuitant.

The value of the benefit is

$$\hat{\mathring{a}}^{(2)}_{x|y} - v^{20} {}_{20}p_{xy}(\hat{\mathring{a}}^{(2)}_{x+20|y+20} - \mathring{a}^{(2)}_{y+20})$$

$$= v^{\frac{1}{4}}(a^{(2)}_y - a^{(2)}_{xy}) - v^{20} {}_{20}p_{xy}\{v^{\frac{1}{4}}(a^{(2)}_{y+20} - a^{(2)}_{x+20:y+20}) - (a^{(2)}_{y+20} + \tfrac{1}{4}\bar{\mathrm{A}}_{y+20})\}$$

$$= v^{\frac{1}{4}}(a_y - a_{xy}) + v^{20} {}_{20}p_{xy}\{a_{y+20}(1 - v^{\frac{1}{4}}) + v^{\frac{1}{4}}a_{x+20:y+20} + \tfrac{1}{4} + \tfrac{1}{4}\mathrm{A}_{y+20}(1+i)^{\frac{1}{4}}\}$$

if $a^{(2)} = a + \frac{1}{4}$ for both single and joint lives.

The value of the payments is

$$P(1+a_{xy:\overline{19}|})$$

whence P may be found by equating the two sides.

18. Write down an expression for the net premium payable by a husband aged 40, to provide an annuity of 1 to his wife aged 30, should she survive him; the premium to be payable quarterly in advance for a period not exceeding 20 years, and the annuity to be entered upon at the death of the husband, and to be payable quarterly with proportion to the death of the widow.

Using Sprague's formula (17)

$$\mathring{a}^{(m)}_{y|x} = v^{\frac{1}{2m}}(a^{(m)}_x - a^{(m)}_{xy})$$

$$= v^{\frac{1}{2m}}(a_x - a_{xy}) \text{ if we assume } a^{(m)} = a + \frac{m-1}{2m}$$

we have $\mathring{a}^{(4)}_{40|30} = v^{\frac{1}{8}}(a_{30} - a_{30:40})$ for the value of the benefit.

And for the payment side

$$_{20}P^{(4)} \times \text{a}^{(4)}_{30:40:\overline{20}|}$$

$$= {}_{20}P^{(4)}\left\{a_{30:40:\overline{20}|} + \tfrac{5}{8}\left(1 - \frac{D_{50:60}}{D_{30:40}}\right)\right\}$$

since $$\text{a}^{(m)}_{xy\overline{n}|} = \text{a}^{(m)}_{xy} - \frac{D_{x+n:y+n}}{D_{xy}}\text{a}^{(m)}_{x+n:y+n}$$

$$= a_{xy} + \frac{m+1}{2m} - \frac{D_{x+n:y+n}}{D_{xy}}\left(a_{x+n:y+n} + \frac{m+1}{2m}\right)$$

$$= a_{xy\overline{n}|} + \frac{m+1}{2m}\left(1 - \frac{D_{x+n:y+n}}{D_{xy}}\right)$$

Hence $${}_{20}P^{(4)} = \frac{v^{\frac{1}{8}}(a_{30} - a_{30:40})}{a_{30:40:\overline{20}|} + \tfrac{5}{8}\left(1 - \frac{D_{50:60}}{D_{30:40}}\right)}$$

19. A, aged 45, wishes his son, aged 15, to receive an annuity-due of £20 on A's attaining 60 years or previous death. Find the

yearly premium at 3 per cent. interest, using the $O^{[NM]}$ Table for the father and the $O^{[M]}$ for the son.

$$\text{Benefit side} = 20a_{45:\overline{14|}|15}$$
$$= 20(a_{15} - a_{15:45:\overline{14|}})$$
$$\text{Payment side} = P(1 + a_{15:45:\overline{14|}})$$

Hence equating the two sides

$$P = \frac{20(a_{15} - a_{15:45:\overline{14|}})}{1 + a_{15:45:\overline{14|}}}$$

An approximation to $a_{15:45:\overline{14|}}$ may be found as follows:—

$$a_{15:45:\overline{14|}} = \frac{1}{P_{15:45:\overline{15|}} + d} - 1$$

But by Lidstone's formula $P_{15:45:\overline{15|}} = P_{15:\overline{15|}} + P_{45:\overline{15|}} - P_{\overline{15|}}$

Now $O^{[M]}a_{[15]:\overline{14|}} = 10{\cdot}899$, $O^{[NM]}a_{[45]:\overline{14|}} = 10{\cdot}192$, and $a_{\overline{14|}} = 11{\cdot}296$

Therefore entering conversion tables, we get

$$P_{[15]:\overline{15|}} = {\cdot}05492,\ P_{[45]:\overline{15|}} = {\cdot}06022, \text{ and } P_{\overline{15|}} = {\cdot}05220$$

whence $P_{[15]:[45]:\overline{15|}} = {\cdot}06294$ approximately.

Entering conversion tables inversely with this value we obtain

$$a_{[15]:[45]:\overline{14|}} = 9{\cdot}861$$

and $$a_{[15]} = 23{\cdot}223$$

Therefore $$P = \frac{20(23{\cdot}223 - 9{\cdot}861)}{10{\cdot}861}$$
$$= 24{\cdot}605, \text{ say } £24,\ 12s.\ 1d.$$

20. There are S persons at present entitled to an annuity of £K per annum each, their ages being respectively $a, b, c \cdot \cdot \cdot l, m, n$. Upon the death of an annuitant, the next on a waiting list steps in. The waiting list consists of T persons, aged respectively $p, q, r \cdot \cdot \cdot x, y, z$. It is required to find the value of the interest of (z) who is Tth on the list.

There are (S + T) persons involved altogether, and (z) will come in only when the (S + T − 1) persons, apart from him, are reduced in number below S. Therefore his chance of getting a payment in any year is the probability that he is alive multiplied by the probability that less than S persons out of (S + T − 1) are alive.

Now the latter probability is equal to the sum of the probabilities that none, that exactly 1, that exactly 2, etc., that exactly (S − 1) persons out of (S + T − 1) are alive

$$= {}_tp_{\overline{abc\ldots lmnpqr\ldots xy}}^{[0]} + {}_tp_{\overline{abc\ldots lmnpqr\ldots xy}}^{[1]}$$

$$+ {}_tp_{\overline{abc\ldots lmnpqr\ldots xy}}^{[2]} + \cdot \cdot \cdot$$

$$+ {}_tp_{\overline{abc\ldots lmnpqr\ldots xy}}^{[S-1]}$$

$$= \frac{1}{1+Z} + \frac{Z}{(1+Z)^2} + \frac{Z^2}{(1+Z)^3} + \cdot \cdot \cdot + \frac{Z^{S-1}}{(1+Z)^S}$$

$$= 1 - \frac{Z^S}{(1+Z)^S}$$

$$= 1 - {}_tp_{\overline{abc\ldots lmnpqr\ldots xy}}^{S}$$

where $abc \ldots lmnpqr \ldots xy$ represents (S + T − 1) persons.

The value of (z)'s interest will therefore be

$$\pounds\mathrm{K}\, \Sigma v^t {}_tp_z (1 - {}_tp_{\overline{abc\ldots lmnpqr\ldots xy}}^{S})$$

CHAPTER XV

Compound Survivorship Annuities and Assurances

1. It has been already pointed out that there are two formulas for the reversionary annuity

$$a_{y|x} = \Sigma v^t{}_tp_x \times {}_{|t}q_y$$

$$\text{and } a_{y|x} = \Sigma v^t \frac{l_{x+t}}{l_x} \frac{d_{y+t-1}}{l_y} (1 + a_{x+t})$$

and these two are identical in value.

Similarly we may express compound survivorship annuities in either of two ways. Thus

$$a^1_{yz|x} = \Sigma v^t{}_tp_x \times {}_{|t}q^1_{yz}$$

for a payment will be made at the end of, say, the tth year provided (x) is then alive, and (y) has died before (z) within the t years. But

$$
\begin{aligned}
{}_{|t}q^1_{yz} &= Q^1_{yz} \times {}_{|t}q_{yz} \text{ approximately.} \\
\text{Therefore } a^1_{yz|x} &= \Sigma v^t{}_tp_x Q^1_{yz} \times {}_{|t}q_{yz} \\
&= Q^1_{yz} a_{yz|x} \\
\text{Also } a_{yz}^{\ 1}{}_{|x} &= \Sigma v^t{}_tp_x({}_{|t}q_{yz} - {}_{|t}q^1_{yz}) \\
&= a_{yz|x} - a^1_{yz|x} \\
\text{And } a_{yz}^{\ 2}{}_{|x} &= \Sigma v^t{}_tp_x \times {}_{|t}q^2_{yz} \\
&= \Sigma v^t{}_tp_x({}_{|t}q_z - {}_{|t}q_{yz}^{\ 1}) \\
&= a_{z|x} - a_{yz}^{\ 1}{}_{|x} \\
&= a_{z|x} - a_{yz|x} + a^1_{yz|x} \\
&= a_x - a_{xz} - a_x + a_{xyz} + a^1_{yz|x} \\
&= a^1_{yz|x} - a_{y|xz}
\end{aligned}
$$

But again $a^{1}_{yz|x} = \Sigma v^t \dfrac{l_{x+t}}{l_x}\dfrac{d_{y+t-1}}{l_y}\dfrac{l_{z+t-\frac{1}{2}}}{l_z}(1+a_{x+t})$

for if (y) die in, say, the tth year—(z) being still alive at (y)'s death—(x), if he survive to the end of that year, will come into possession of an annuity-due for the remainder of life.

By this method $a_{y\overset{1}{z}|x}$ takes a similar form, and can then be reduced as before to $a_{yz|x} - a^{1}_{yz|x}$

$$\text{And } a_{y\overset{2}{z}|x} = \Sigma v^t \frac{l_{x+t}}{l_x}\frac{d_{z+t-1}}{l_z}\frac{l_y - l_{y+t-\frac{1}{2}}}{l_y}(1+a_{x+t})$$

$$= a_{z|x} - a_{y\overset{1}{z}|x} \text{ as before.}$$

$$\text{Also } \mathrm{A}^{\,3}_{\underset{1}{x}yz} = \Sigma v^t \frac{l_z - l_{z+t-\frac{1}{2}}}{l_z}\frac{d_{y+t-1}}{l_y}\frac{l_{x+t-\frac{1}{2}}}{l_x}\left(1 - \frac{l_{x+t}}{l_{x+t-\frac{1}{2}}}\,da_{x+t}\right)$$

$$= \Sigma v^t \frac{l_z - l_{z+t-\frac{1}{2}}}{l_z}\frac{d_{y+t-1}}{l_y}\frac{l_{x+t-\frac{1}{2}}}{l_x}$$

$$- d\Sigma v^t \frac{l_{x+t}}{l_x}\frac{d_{y+t-1}}{l_y}\frac{l_z - l_{z+t-\frac{1}{2}}}{l_z}(1+a_{x+t})$$

$$= \mathrm{A}^{\,2}_{\underset{1}{x}yz} - da^{2}_{yz|x}$$

which is *Text Book* formula (6).

2. To find the annual premium for $a^{1}_{yz|x}$, we must divide the benefit by $(1+a_{xyz})$. (y) must be medically examined; for it is on his death the benefit starts, and (x)'s or (z)'s death earlier relieves the office of a liability.

The annual premium for $a^{2}_{yz|x}$ is found by dividing the benefit by $(1+a_{xy})$. Probably (y) and (z) should both be medically examined.

The annual premium for $\mathrm{A}^{\,3}_{\underset{1}{x}yz}$ is payable at least so long as both (x) and (y) are alive, and also for the period of (x)'s life after the death of (y), but only provided (y) died after (z). Therefore we must divide the benefit by

$$1 + a_{xy} + a^{2}_{yz|x} = 1 + a_{xyz} + a_{y\overset{1}{z}|x}.$$

Probably (x), (y), and (z) should all be examined, but in this case and the preceding one much will depend on the respective ages of the lives.

3. The interpretation of the following symbols should be noted—

$a_{\overline{n}|wxy\ldots(m)}^{\;\;r}$ is an annuity payable so long as at least r of the m statuses $\overline{n}|$, w, x, y, etc., survive. It is therefore equivalent to a temporary annuity for n years on $(r-1)$ lives out of $(m-1)$, together with an annuity deferred n years on r out of $(m-1)$, that is $_{|n}a_{\overline{wxy\ldots(m-1)}}^{\;r-1} + {}_{n|}a_{\overline{wxy\ldots(m-1)}}^{\;r}$

$a_{\overline{y\overline{n}|}|x}$ is an annuity to (x) of which the first payment is to be made at the end of the year in which (y) dies, but in no case is a payment to be made till $(n+1)$ years from the present time.

$a_{\overline{xy}:\overline{n}||\overline{t}|}^{\;1}$ is a reversionary annuity to commence on the failure of the joint lives (x) and (y) if that event occurs within n years, but in no case are payments to extend beyond t years from the present.

$A_{\overline{xy}:z \atop 1:3}^{\left(\frac{t}{2}\right)}$ is an assurance payable at the end of the $\frac{2}{t}$ part of the year in which the survivor of (x) and (y) dies (that is, on the average $\frac{1}{t}$ of a year after such death), provided (z) die either before or after both the others.

$a_x^{(\frac{1}{5})}$. Just as $a_x^{(m)}$ is an annuity to (x) payable in instalments of $\frac{1}{m}$ at the end of each $\frac{1}{m}$ of a year, so $a_x^{(\frac{1}{5})}$ is an annuity to (x) payable in instalments of 5 at the end of each period of 5 years.

That is

$$a_x^{(\frac{1}{5})} = \frac{5D_{x+5}+5D_{x+10}+5D_{x+15}+\cdots}{D_x}$$

$$= \frac{(D_{x+3}+D_{x+4}+D_{x+5}+D_{x+6}+D_{x+7})+(D_{x+8}+\cdots+D_{x+12})+\cdots}{D_x}$$

$$= \frac{N_{x+2}}{D_x}$$

if we assume $5D_{x+n} = D_{x+n-2}+D_{x+n-1}+D_{x+n}+D_{x+n+1}+D_{x+n+2}$

EXAMPLES

1. Find the premium for an assurance payable on the death of the longest liver of A, B, and C, aged x, y, and z respectively: should A die first the premium to be reduced 50 per cent., and should C survive A and B the premium to cease.

The benefit side $= A_{\overline{xyz}}$

And the payment side can be expressed in either of the two forms

$$P(a_{\overline{xyz}} - \tfrac{1}{2}a_{x|\overline{yz}} - \tfrac{1}{2}a_{\overline{xy}|z} + \tfrac{1}{2}a^{2}_{xz|y})$$

and $$P(a_x + \tfrac{1}{2}a_{x|yz} + \tfrac{1}{2}a_{x\overset{2}{z}|y} + a^{2}_{xz|y})$$

whence P may be found.

2. Give a formula for the present value of £1 receivable on the death of a person aged 50, provided another person now aged 20 has then either died or attained age 40.

This assurance, being payable on the death of (50), provided either (20) has previously died or 20 years certain have elapsed, may be represented by

$$A^{2:3}_{50:20:\overline{20}|} = A_{50} - A_{\overset{1}{50}:20:\overline{20}|}$$

3. Determine $A^{3:4}_{w:x\underset{2}{y}\underset{1}{z}}$

This is the value of an assurance on the death of (w) provided he die either third or fourth of the four lives (w), (x), (y), and (z); and provided (z) and (y) have died first and second of the four respectively.

$$\text{Now} \quad A^{3:4}_{w:x\underset{2}{y}\underset{1}{z}} = A_{wx\overset{2}{y}\underset{1}{z}} - da_{x\overset{2}{y}\underset{1}{z}|w}$$

$$= A_{wx\overset{1}{y}} - A_{wxy\overset{1}{z}} - da_{x\overset{2}{y}\underset{1}{z}|w}$$

$$\text{But} \quad a_{x\overset{2}{y}\underset{1}{z}|w} = \Sigma v^t {}_tp_w \times |_tq_{x\overset{2}{y}\underset{1}{z}}$$

$$= \Sigma v^t {}_tp_w(|_tq_{x\overset{1}{y}} - |_tq_{xy\overset{1}{z}})$$

$$= a_{x\overset{1}{y}|w} - a_{xy\overset{1}{z}|w}$$

$$= Q_{x\overset{1}{y}}a_{xy|w} - Q_{xy\overset{1}{z}}a_{xyz|w} \text{ approximately.}$$

$$\text{Hence} \quad A^{3:4}_{w:x\underset{2}{y}\underset{1}{z}} = A_{wx\overset{1}{y}} - A_{wxy\overset{1}{z}} - d(Q_{x\overset{1}{y}}a_{xy|w} - Q_{xy\overset{1}{z}}a_{xyz|w})$$

4. Find the value of

$$(a)\quad A^{1:3}_{\overline{xy}:z}$$

$$(b)\quad A^{2}_{x:\overline{y(\overline{t|})}}$$

$$(c)\quad A_{\overline{xy}:\underset{1:3}{z}}$$

$$(d)\quad A^{1:3}_{x:yz}$$

$$(e)\quad A_{z|\overline{xy}}$$

(*a*) This is an assurance payable (1) if (x) die first, (2) if (x) die third, (z) having died first, (3) if (y) die first, or (4) if (y) die third, (z) having died first. In symbols

$$A_{\overset{1}{x}yz} + A_{\overset{3}{x}y\underset{1}{z}} + A_{x\overset{1}{y}z} + A_{x\overset{3}{y}\underset{1}{z}}$$

(*b*) This is an assurance payable on the death of (x) if he die more than t years after (y). In symbols

$$A_x - A^{1}_{x:\overline{y(\overline{t|})}}$$

(*c*) This is an assurance payable (1) on the death of (x) after (y), if (z) has died before (y) or if (z) is still alive; (2) on the death of (y) after (x), if (z) has died before (x), or if (z) is still alive. In symbols

$$A_{\overset{3}{x}y\underset{1}{z}} + A_{\overset{2}{x}y\underset{1}{z}} + A_{x\overset{3}{y}\underset{1}{z}} + A_{x\overset{2}{y}\underset{1}{z}}$$

(*d*) This is an assurance payable on the death of (x) if he die first or last of the three lives. In symbols

$$A_x - A_{\overset{2}{x}yz} = A_x - A_{\overset{1}{x}y} - A_{\overset{1}{x}z} + 2A_{\overset{1}{x}yz}$$

(*e*) This is an assurance payable on the death of the survivor of (x) and (y) if that should happen before the death of (z). The alternative symbol is $A_{\overline{xy}:\underset{3}{z}}$

5. Express $\bar{A}^{2:3:4}_{w:xy\underset{1}{z}}$ and $\bar{A}^{3:4}_{w:xy\underset{21}{z}}$ in formulas for summation not involving the use of the integral calculus.

$$\bar{A}^{2:3:4}_{\underset{1}{w:xyz}} = \Sigma_{t=1}^{t=\infty} v^{t-\frac{1}{2}} \frac{d_{z+t-1}}{l_z} \frac{l_{w+t-\frac{1}{2}} l_{x+t-\frac{1}{2}} l_{y+t-\frac{1}{2}}}{l_w l_x l_y} \bar{A}_{w+t-\frac{1}{2}}$$

$$\bar{A}^{3:4}_{\underset{21}{w:xyz}} = \Sigma_{t=1}^{t=\infty} v^{t-\frac{1}{2}} \frac{l_z - l_{z+t-\frac{1}{2}}}{l_z} \frac{d_{y+t-1}}{l_y} \frac{l_{w+t-\frac{1}{2}} l_{x+t-\frac{1}{2}}}{l_w l_x} \bar{A}_{w+t-\frac{1}{2}}$$

6. The present holder of a title of nobility is aged w. It is desired to effect an assurance payable on the death of his wife aged x, provided that during her lifetime, the heir aged y having died, the next heir aged z shall have succeeded to the title. Give a formula for the single premium.

To fulfil the conditions both (w) and (y) must die before both (x) and (z), but it is immaterial whether (x) dies before or after (z). The single premium therefore is

$$A^{\quad 2:3}_{\underset{1}{\overline{wy}}:x:z} = A^{3:4}_{\underset{1\ \ 2}{w:x:y:z}} + A^{3:4}_{\underset{2\ \ 1}{w:x:y:z}}$$

It may be most easily expressed as an integral, as follows:—

$$\bar{A} = \int_0^\infty v^t {}_tp_{xz}\{(1 - {}_tp_y){}_tp_w\mu_{w+t} + (1 - {}_tp_w){}_tp_y\mu_{y+t}\}\bar{A}_{x+t}dt$$

7. Calculate by the *Text Book* Mortality Table the value of the following formula, using 4 per cent. interest:—

$$500\, A^1_{70:70} - 40a^1_{70:70|50}$$

$$\begin{aligned} 500\, A^1_{70:70} - 40\, a^1_{70:70|50} & \\ &= 500 \times \tfrac{1}{2}A_{70:70} - 40 \times \tfrac{1}{2}a_{70:70|50} \end{aligned}$$

(since by formula (4) of this chapter of the *Text Book* $a^1_{70:70|50} = Q^1_{70:70}a_{70:70|50} = \frac{1}{2}a_{70:70|50}$)

$$= 250\, A_{70:70} - 20(a_{50} - a_{50:70:70})$$

$$\begin{aligned} \text{We have } A_{70:70} &= 1 - d(1 + a_{70:70}) \\ &= 1 - {\cdot}03846(1 + 4{\cdot}054) \\ &= 1 - {\cdot}19438 \\ &= {\cdot}80562 \end{aligned}$$

And $$a_{50} = 12{\cdot}522$$

$$a_{50:70:70} = a_{x:x:x} \text{ where } \mu_{50}+\mu_{70}+\mu_{70} = 3\mu_x$$

By the Table of Uniform Seniority

$$c^{50}+c^{70}+c^{70} = 3c^{66\cdot4}$$

$$a_{66\cdot4:66\cdot4:66\cdot4} = 3{\cdot}850 - {\cdot}4\times{\cdot}228$$

$$= 3{\cdot}759.$$

Therefore $500\,A^{1}_{70:70} - 40\,a^{1}_{70:70|50}$

$$= 250\times{\cdot}80562 - 20(12{\cdot}522 - 3{\cdot}759)$$

$$= 201{\cdot}405 - 175{\cdot}260$$

$$= 26{\cdot}145.$$

CHAPTER XVI

Commutation Columns, Varying Benefits, and Returns of Premiums

1. In addition to the expressions derived in *Text Book,* Articles 9 to 14, the following should be carefully examined. It will be found that these or similar expressions are very frequently required in Chapter XVIII. in connection with valuation by the retrospective method, and it is essential that the principles upon which they are founded should be thoroughly understood.

$$\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}} = \frac{l_x(1+i)^n + l_{x+1}(1+i)^{n-1} + \cdots + l_{x+n-1}(1+i)}{l_{x+n}}$$

This represents the accumulations to the end of n years of an annuity-due on (x) for that period. It should be noticed that it is greater than $(1+i)s_{\overline{n}|}$; for each value of l in the numerator is greater than l_{x+n} in the denominator, and the whole expression is accordingly greater than $(1+i)^n+(1+i)^{n-1}+\cdots+(1+i)$, which is the value of $(1+i)s_{\overline{n}|}$.

$$\frac{N_{x-1}-N_{x+t-1}}{D_{x+n}} = \frac{l_x(1+i)^n + l_{x+1}(1+i)^{n-1} + \cdots + l_{x+t-1}(1+i)^{n-t+1}}{l_{x+n}}$$

$$\frac{M_x - M_{x+n}}{D_{x+n}} = \frac{d_x(1+i)^{n-1} + d_{x+1}(1+i)^{n-2} + \cdots + d_{x+n-1}}{l_{x+n}}$$

$$\frac{M_{x+t} - M_{x+n}}{D_{x+n}} = \frac{d_{x+t}(1+i)^{n-t-1} + d_{x+t+1}(1+i)^{n-t-2} + \cdots + d_{x+n-1}}{l_{x+n}}$$

$$\frac{R_x - R_{x+n} - nM_{x+n}}{D_{x+n}} = \frac{d_x(1+i)^{n-1} + 2d_{x+1}(1+i)^{n-2} + \cdots + nd_{x+n-1}}{l_{x+n}}$$

$$\frac{R_x - R_{x+t} - tM_{x+t}}{D_{x+n}} = \frac{d_x(1+i)^{n-1} + 2d_{x+1}(1+i)^{n-2} + \cdots + td_{x+t-1}(1+i)^{n-t}}{l_{x+n}}$$

$$\frac{R_x - R_{x+t} - tM_{x+n}}{D_{x+n}} = \frac{1}{l_{x+n}}\Big[d_x(1+i)^{n-1} + 2d_{x+1}(1+i)^{n-2} + \cdots$$
$$+ t\{d_{x+t-1}(1+i)^{n-t} + d_{x+t}(1+i)^{n-t-1} + \cdots + d_{x+n-1}\}\Big]$$

2. The following is probably a simpler method of obtaining the values of varying and increasing annuities and assurances.

$$(\mathrm{v}a)_x = \frac{k\mathrm{N}_x \pm h(\mathrm{N}_{x+1} + \mathrm{N}_{x+2} + \mathrm{N}_{x+3} + \cdot \cdot \cdot)}{\mathrm{D}_x}$$

$$= \frac{k\mathrm{N}_x \pm h\mathrm{S}_{x+1}}{\mathrm{D}_x}$$

When $k = h = 1$

$$(\mathrm{I}a)_x = \frac{\mathrm{S}_x}{\mathrm{D}_x}$$

$$(\mathrm{vA})_x = \frac{k\mathrm{M}_x \pm h(\mathrm{M}_{x+1} + \mathrm{M}_{x+2} + \mathrm{M}_{x+3} + \cdot \cdot \cdot)}{\mathrm{D}_x}$$

$$= \frac{k\mathrm{M}_x \pm h\mathrm{R}_{x+1}}{\mathrm{D}_x}$$

When $k = h = 1$

$$(\mathrm{IA})_x = \frac{\mathrm{R}_x}{\mathrm{D}_x}$$

$$(\mathrm{v}_{\overline{n}|}a)_x = \frac{k\mathrm{N}_x \pm h(\mathrm{N}_{x+1} + \mathrm{N}_{x+2} + \cdot \cdot \cdot + \mathrm{N}_{x+n-1})}{\mathrm{D}_x}$$

$$= \frac{k\mathrm{N}_x \pm h(\mathrm{S}_{x+1} - \mathrm{S}_{x+n})}{\mathrm{D}_x}$$

When $k = h = 1$

$$(\mathrm{I}_{\overline{n}|}a)_x = \frac{\mathrm{S}_x - \mathrm{S}_{x+n}}{\mathrm{D}_x}$$

Also $$(\mathrm{v}_{\overline{n}|}\mathrm{A})_x = \frac{k\mathrm{M}_x \pm h(\mathrm{M}_{x+1} + \mathrm{M}_{x+2} + \cdot \cdot \cdot + \mathrm{M}_{x+n-1})}{\mathrm{D}_x}$$

$$= \frac{k\mathrm{M}_x \pm h(\mathrm{R}_{x+1} - \mathrm{R}_{x+n})}{\mathrm{D}_x}$$

When $k = h = 1$

$$(\mathrm{I}_{\overline{n}|}\mathrm{A})_x = \frac{\mathrm{R}_x - \mathrm{R}_{x+n}}{\mathrm{D}_x}$$

$(va)_{\overline{xn}|}$

$$= \frac{k(N_x - N_{x+n}) \pm h\{(N_{x+1} - N_{x+n}) + (N_{x+2} - N_{x+n}) + \cdots + (N_{x+n-1} - N_{x+n})\}}{D_x}$$

$$= \frac{k(N_x - N_{x+n}) \pm h\{S_{x+1} - S_{x+n} - (n-1)N_{x+n}\}}{D_x}$$

When $k = h = 1$

$$(Ia)_{\overline{xn}|} = \frac{S_x - S_{x+n} - nN_{x+n}}{D_x}$$

Also $(vA)^1_{\overline{xn}|}$

$$= \frac{k(M_x - M_{x+n}) \pm h\{(M_{x+1} - M_{x+n}) + (M_{x+2} - M_{x+n}) + \cdots + (M_{x+n-1} - M_{x+n})\}}{D_x}$$

$$= \frac{k(M_x - M_{x+n}) \pm h\{R_{x+1} - R_{x+n} - (n-1)M_{x+n}\}}{D_x}$$

When $k = h = 1$

$$(IA)^1_{\overline{xn}|} = \frac{R_x - R_{x+n} - nM_{x+n}}{D_x}$$

3. So long as the S_x and R_x columns are supplied, the working out of increasing benefits by these formulas is therefore an easy matter. But in cases where these commutation columns are not available, a method which has been described by Mr Lidstone (*J. I. A.*, xxxi. 68) may be used. The proof upon which the method rests is as follows:—

Let B_x be a benefit of any nature dependent on the life (x), and expressed by $vp_1 + v^2p_2 + v^3p_3 + \cdot \cdot \cdot$, where p_1, p_2, p_3, etc., are the probabilities of a payment being made at the end of the first, second, third, etc., years.

$$\text{Then} \quad \frac{d}{di} B_x = \frac{d}{dv} B_x \times \frac{dv}{di}$$

$$= (p_1 + 2vp_2 + 3v^2p_3 + \cdot \cdot \cdot)(-v^2)$$

$$= -(v^2p_1 + 2v^3p_2 + 3v^4p_3 + \cdot \cdot \cdot)$$

$$\text{Therefore} \quad -(1+i)\frac{d}{di} B_x = vp_1 + 2v^2p_2 + 3v^3p_3 + \cdot \cdot \cdot$$

$$= (IB)_x$$

where $(IB)_x$ is a benefit dependent on the same probabilities as B_x, but increasing by 1 per annum throughout.

$$\text{But} \quad \frac{d}{di} B_x = \frac{\Delta B_x - \frac{1}{2}\Delta^2 B_x}{\Delta i} \text{ approximately.}$$

$$\text{Hence} \quad (IB)_x = -(1+i)\frac{\Delta B_x - \frac{1}{2}\Delta^2 B_x}{\Delta i} \text{ approximately.}$$

This formula is perfectly general and applicable to any type of benefit which increases uniformly.

For example, let it be required to find the value of $(IA)_{45}$ by the *Text Book* Table at $3\frac{1}{2}$ per cent.

$$\begin{aligned}(IA)_{45} &= -1\cdot035\,\frac{(A_{45(4\%)} - A_{45(3\frac{1}{2}\%)}) - \frac{1}{2}(A_{45(4\frac{1}{2}\%)} - 2A_{45(4\%)} + A_{45(3\frac{1}{2}\%)})}{\cdot005} \\ &= -207\{(\cdot42692 - \cdot46889) - \tfrac{1}{2}(\cdot39003 - \cdot85384 + \cdot46889)\} \\ &= 207(\cdot04197 + \cdot00254) \\ &= 9\cdot21357\end{aligned}$$

4. The difficulties and dangers attending the practice (recommended in *Text Book*, Article 27) of omitting the denominator in writing benefit and payment sides are such as to outweigh any slight saving of trouble.

Theoretically, such expressions as $(M_x - M_{x+n})$, $(N_{x-1} - N_{x+n-1})$, etc., have no meaning as they stand (*vide Text Book*, Chapter VII., Article 9), and in practical use they will have different senses according to the particular denominator used with them. Therefore until the proper denominator is fixed the proper sense cannot be ascertained. It is only after supplying denominators to both that benefit and payment sides can be examined and compared to check their accuracy. Further, where a second life comes into the problem, and the denominator is omitted everywhere, the fact may be overlooked that, *e.g.*, D_x is the denominator for part of the problem and D_{xy} for the remainder, and thereby serious error may be introduced.

5. To find the value of the temporary benefits mentioned in *Text Book*, Article 46, we must stop at the nth value of u, that is, u_{n-1}, which is equal to

$$u_0 + (n-1)\Delta u_0 + \frac{(n-1)(n-2)}{\lfloor 2}\Delta^2 u_0 + \cdot\;\cdot\;\cdot$$

Then we shall have

$$(va)_{\overline{xn}|} = \frac{1}{D_x}\Big[(N_x - N_{x+n})u_0 + \Big\{S_{x+1} - S_{x+n} - (n-1)N_{x+n}\Big\}\Delta u_0$$
$$+\Big\{\Sigma S_{x+2} - \Sigma S_{x+n} - (n-2)S_{x+n} - \frac{(n-1)(n-2)}{2}N_{x+n}\Big\}\Delta^2 u_0$$
$$+ \cdot \cdot \cdot \Big]$$

$$(vA)^1_{\overline{xn}|} = \frac{1}{D_x}\Big[(M_x - M_{x+n})u_0 + \Big\{R_{x+1} - R_{x+n} - (n-1)M_{x+n}\Big\}\Delta u_0$$
$$+\Big\{\Sigma R_{x+2} - \Sigma R_{x+n} - (n-2)R_{x+n} - \frac{(n-1)(n-2)}{2}M_{x+n}\Big\}\Delta^2 u_0$$
$$+ \cdot \cdot \cdot \Big]$$

To obtain $(v_{\overline{n}|}a)_x$ and $(v_{\overline{n}|}A)_x$ we need only omit from these formulas the terms which cut off the whole benefit at the end of n years, but retain the terms which cut off the increase merely. Thus

$$(v_{\overline{n}|}a)_x = \frac{1}{D_x}\Big[N_x u_0 + (S_{x+1} - S_{x+n})\Delta u_0$$
$$+\{\Sigma S_{x+2} - \Sigma S_{x+n} - (n-2)S_{x+n}\}\Delta^2 u_0$$
$$+ \cdot \cdot \cdot \Big]$$

$$(v_{\overline{n}|}A)_x = \frac{1}{D_x}\Big[M_x u_0 + (R_{x+1} - R_{x+n})\Delta u_0$$
$$+\{\Sigma R_{x+2} - \Sigma R_{x+n} - (n-2)R_{x+n}\}\Delta^2 u_0$$
$$+ \cdot \cdot \cdot \Big]$$

6. The warning contained in *Text Book*, Article 98, should be carefully noted, as the error presents itself in different forms. For example, if it is desired to have a table giving the annual premiums for pure endowments, one-half of the premiums to be returnable in the event of death before the expiry of the term, it is incorrect to take the arithmetic mean of the premiums for pure endowments with and without return respectively. The correct office premium for the new benefit is $P(1+\kappa)+c$ where

$$P = \frac{D_{x+n} + \frac{1}{2}c(R_x - R_{x+n} - nM_{x+n})}{N_{x-1} - N_{x+n-1} - \frac{1}{2}(1+\kappa)(R_x - R_{x+n} - nM_{x+n})}$$

while the proposed office premium is $\pi(1+\kappa)+c$ where

$$\pi = \frac{1}{2}\left\{\frac{D_{x+n}}{N_{x-1}-N_{x+n-1}} + \frac{D_{x+n}+c(R_x - R_{x+n} - nM_{x+n})}{N_{x-1}-N_{x+n-1}-(1+\kappa)(R_x - R_{x+n} - nM_{x+n})}\right\},$$

and these two are not equal.

The explanation is that if (x) were to die within the n years, having taken out a policy at this proposed premium, the office will return only one-half of the premiums paid; but if, on the other hand, he had effected two policies, one with and the other without return, each for one-half of the sum assured, the office would have to return the whole premiums under the former policy which obviously are greater than the mean of the premiums under both policies. In accepting the contract at the proposed premium the assured is therefore allowing himself to be overcharged.

7. We proceed to discuss some practical problems not dealt with in the *Text Book*.

It sometimes happens that (x) for some reason will not be accepted by an office at the normal premium for his then age. He, however, refuses to pay the premium for an older age which they wish to charge him, but consents to his policy at the normal premium bearing the condition that the sum assured will be paid under deduction of a certain sum in the event of his dying within t years and in full on death thereafter, t being usually fixed at the expectation of life of (x). It is required to obtain a formula to determine the amount of this "Contingent Debt."

First, let us assume the debt to be constant during the t years, and equal to $\mathbf{X}$.

Taking the life at his assumed or rated-up age we see that the value of the premiums which the office should receive is

$$P_{x+n}(1+a_{x+n})$$

But they are to receive only $P_x(1+a_{x+n})$. Therefore they lose premiums to the value of

$$(P_{x+n}-P_x)(1+a_{x+n})$$

Now the present value of the debt is

$$\mathbf{X}\,A_{\overset{1}{x+n}:\overline{t|}}$$

Therefore equating and solving for $\mathbf{X}$ we have

$$\mathbf{X} = \frac{(P_{x+n}-P_x)(1+a_{x+n})}{A_{\overset{1}{x+n}:\overline{t|}}}$$

Again, assume the debt to commence at $t\mathbf{X}$ and decrease by $\mathbf{X}$ each year till it disappears at the end of t years.

As before, the value of the premiums which the office forgoes is

$$(P_{x+n} - P_x)(1 + a_{x+n}) = \frac{(P_{x+n} - P_x)N_{x+n-1}}{D_{x+n}}$$

The present value of the debt is now

$$\frac{t\mathbf{X}M_{x+n} - \mathbf{X}(R_{x+n+1} - R_{x+n+t+1})}{D_{x+n}}$$

Equating and solving we have

$$\mathbf{X} = \frac{(P_{x+n} - P_x)N_{x+n-1}}{tM_{x+n} - (R_{x+n+1} - R_{x+n+t+1})}$$

In this investigation the damaged life is assumed to be a normal life aged $(x+n)$, and the extra rates of mortality for successive years are accordingly as follows:—

Year of Duration.	Extra rate of Mortality.
1	$q_{x+n} - q_x$
2	$q_{x+n+1} - q_{x+1}$
3	$q_{x+n+2} - q_{x+2}$
etc.	etc.

It will be found that this extra mortality is at first small and slowly increasing, but becomes great in the later years of duration; and this is a comparatively uncommon form of extra risk. Also from the nature of the contract, the contingent-debt scheme should be specially applicable to the case where the extra risk is at first large and afterwards decreasing. Accordingly, the method of fixing the amount of the debt is open to criticism in these respects.

8. We here give the methods of ascertaining the premiums when they are loaded for bonuses in addition to the sum assured of 1.

(*a*). **Uniform Reversionary Bonus.**—The problem is to find the annual premium for an assurance of 1, to increase by $5b$ every 5 years, with an interim bonus of b' in respect of each premium

paid since the commencement or since the last increase, in the event of death within any quinquennial period.

$$\text{Benefit side} = \frac{M_x}{D_x} + 5b\frac{M_{x+5} + M_{x+10} + \cdot\cdot\cdot}{D_x}$$
$$+ b'\frac{R_x - 5(M_{x+5} + M_{x+10} + \cdot\cdot\cdot)}{D_x}$$

$$\text{Payment side} = \pi\frac{N_{x-1}}{D_x}$$

whence we may obtain π.

If no interim bonus is to be given for the first quinquennium, the benefit side becomes

$$\frac{M_x}{D_x} + 5b\frac{M_{x+5} + M_{x+10} + \cdots}{D_x} + b'\frac{R_{x+5} - 5(M_{x+10} + M_{x+15} + \cdots)}{D_x}$$

while the payment side remains

$$\pi\frac{N_{x-1}}{D_x}$$

and the new value of π may be found.

If $b' = b$ the benefit side in the first case becomes

$$\frac{M_x}{D_x} + b\frac{R_x}{D_x}$$

and the payment side being

$$\pi\frac{N_{x-1}}{D_x}$$

$$\pi = \frac{M_x + bR_x}{N_{x-1}}$$

While if $b' = b$, and no interim bonus be given for the first quinquennium, the benefit side becomes

$$\frac{M_x}{D_x} + b\frac{5M_{x+5} + R_{x+5}}{D_x}$$

$$\text{Payment side} = \pi\frac{N_{x-1}}{D_x}$$

and in this case $\pi = \dfrac{M_x + b(5M_{x+5} + R_{x+5})}{N_{x-1}}$

For an endowment assurance with a similar bonus, we have

$$\text{Benefit side} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x} + 5b\left(\frac{M_{x+5} - M_{x+n} + D_{x+n}}{D_x}\right.$$

$$+\frac{M_{x+10} - M_{x+n} + D_{x+n}}{D_x} + \cdots + \frac{M_{x+n-5} - M_{x+n} + D_{x+n}}{D_x} + \left.\frac{M_{x+n} - M_{x+n} + D_{x+n}}{D_x}\right)$$

$$+b'\frac{(R_x - R_{x+n}) - 5(M_{x+5} + M_{x+10} + \cdots + M_{x+n})}{D_x}$$

$$= \frac{M_x - M_{x+n} + D_{x+n}}{D_x} + b\frac{5(M_{x+5} + M_{x+10} + \cdots + M_{x+n}) + n(D_{x+n} - M_{x+n})}{D_x}$$

$$+b'\frac{(R_x - R_{x+n}) - 5(M_{x+5} + M_{x+10} + \cdots + M_{x+n})}{D_x}$$

$$\text{Payment side} = \pi\frac{N_{x-1} - N_{x+n-1}}{D_x}$$

and π may at once be obtained.

If no interim bonus is to be given for the first five years, the last term on the benefit side becomes

$$b'\frac{(R_{x+5} - R_{x+n}) - 5(M_{x+10} + M_{x+15} + \cdots + M_{x+n})}{D_x}$$

If $b' = b$, the benefit side becomes in the first case

$$\frac{M_x - M_{x+n} + D_{x+n}}{D_x} + b\frac{R_x - R_{x+n} - nM_{x+n} + nD_{x+n}}{D_x}$$

and in the second case

$$\frac{M_x - M_{x+n} + D_{x+n}}{D_x} + b\frac{5M_{x+5} + R_{x+5} - R_{x+n} - nM_{x+n} + nD_{x+n}}{D_x}$$

Also, the form of the payment side remaining throughout

$$\pi\frac{N_{x-1} - N_{x+n-1}}{D_x}$$

the various values of π may be deduced by equating and solving.

(*b*). **Compound Reversionary Bonus.**—To find the annual premium for an assurance of 1, to increase by $5b$ per unit every 5 years, calculated on the sum assured and existing increases, with an interim bonus of b' per unit in respect of each premium paid since the commencement or since the last increase, also

calculated on the sum assured and existing increases, in the event of death within any quinquennial period.

$$\text{Benefit side} = \frac{1}{D_x}\Big\{(M_x - M_{x+5}) + (1+5b)(M_{x+5} - M_{x+10}) + (1+5b)^2(M_{x+10} - M_{x+15}) + \cdot\cdot\cdot\Big\}$$
$$+ \frac{b'}{D_x}\Big\{(R_x - R_{x+5} - 5M_{x+5}) + (1+5b)(R_{x+5} - R_{x+10} - 5M_{x+10}) + (1+5b)^2(R_{x+10} - R_{x+15} - 5M_{x+15}) + \cdot\cdot\cdot\Big\}$$

$$\text{Payment side} = \pi\frac{N_{x-1}}{D_x}$$

and π may be obtained by equating and solving.

If no interim bonus is to be given for the first five years, the first term of the second part of the benefit side will be omitted, and it will then read—

$$\frac{1}{D_x}\Big\{(M_x - M_{x+5}) + (1+5b)(M_{x+5} - M_{x+10}) + (1+5b)^2(M_{x+10} - M_{x+15}) + \cdot\cdot\cdot\Big\}$$
$$+ \frac{b'}{D_x}\Big\{(1+5b)(R_{x+5} - R_{x+10} - 5M_{x+10}) + (1+5b)^2(R_{x+10} - R_{x+15} - 5M_{x+15}) + \cdot\cdot\cdot\Big\}$$

If $b' = b$ the benefit side in the first case becomes

$$\frac{1}{D_x}\Big\{M_x + b(R_x - R_{x+5}) + b(1+5b)(R_{x+5} - R_{x+10}) + \cdot\cdot\cdot\Big\}$$

and in the second case

$$\frac{1}{D_x}\Big\{M_x + 5bM_{x+5} + b(1+5b)(R_{x+5} - R_{x+10}) + b(1+5b)^2(R_{x+10} - R_{x+15}) + \cdot\cdot\cdot\Big\}$$

The form of the payment side is constant, and therefore the several values of π for these benefits may be obtained.

To find the annual premium for an endowment assurance with a similar bonus, we have

Benefit side

$$= \frac{1}{D_x}\Big\{(M_x - M_{x+5}) + (1+5b)(M_{x+5} - M_{x+10}) + (1+5b)^2(M_{x+10} - M_{x+15}) + \cdots$$
$$+ (1+5b)^{\frac{n-5}{5}}(M_{x+n-5} - M_{x+n}) + (1+5b)^{\frac{n}{5}}D_{x+n}\Big\}$$
$$+ \frac{b'}{D_x}\Big\{(R_x - R_{x+5} - 5M_{x+5}) + (1+5b)(R_{x+5} - R_{x+10} - 5M_{x+10}) + \cdots$$
$$+ (1+5b)^{\frac{n-5}{5}}(R_{x+n-5} - R_{x+n} - 5M_{x+n})\Big\}$$

$$\text{Payment side} = \pi\frac{N_{x-1} - N_{x+n-1}}{D_x}$$

whence we may obtain π.

If no interim bonus is to be given for the first five years the benefit side becomes

$$\frac{1}{D_x}\Big\{(M_x - M_{x+5}) + (1+5b)(M_{x+5} - M_{x+10}) + \cdots$$
$$+ (1+5b)^{\frac{n-5}{5}}(M_{x+n-5} - M_{x+n}) + (1+5b)^{\frac{n}{5}}D_{x+n}\Big\}$$
$$+ \frac{b'}{D_x}\Big\{(1+5b)(R_{x+5} - R_{x+10} - 5M_{x+10}) + \cdots$$
$$+ (1+5b)^{\frac{n-5}{5}}(R_{x+n-5} - R_{x+n} - 5M_{x+n})\Big\}$$

If $b' = b$, the benefit side in the first case becomes

$$\frac{1}{D_x}\Big\{M_x + b(R_x - R_{x+5}) + b(1+5b)(R_{x+5} - R_{x+10}) + \cdots$$
$$+ b(1+5b)^{\frac{n-5}{5}}(R_{x+n-5} - R_{x+n}) + (1+5b)^{\frac{n}{5}}(D_{x+n} - M_{x+n})\Big\}$$

and in the second case

$$\frac{1}{D_x}\Big\{M_x + 5bM_{x+5} + b(1+5b)(R_{x+5} - R_{x+10}) + \cdots$$
$$+ b(1+5b)^{\frac{n-5}{5}}(R_{x+n-5} - R_{x+n}) + (1+5b)^{\frac{n}{5}}(D_{x+n} - M_{x+n})\Big\}$$

Then, the form of the payment side remaining unchanged, the several values of π may be obtained.

Where $b' = b = \cdot 01$ it may be shown that the single premium at 4 per cent. for an assurance of 1 with that compound rever-

sionary bonus is approximately equal to the single premium at 3 per cent. for an assurance of 1. For ease in working, let it be assumed that the bonus is compounded yearly. This will have the effect of increasing the value of the benefit, which may then be expressed

$$\frac{1}{l_x}\left\{\frac{1\cdot01}{1\cdot04}\,d_x+\left(\frac{1\cdot01}{1\cdot04}\right)^2 d_{x+1}+\left(\frac{1\cdot01}{1\cdot04}\right)^3 d_{x+2}+\,\ldots\right\}$$

Now $\frac{1\cdot01}{1\cdot04}=\frac{1}{1\cdot03}$ approximately, and if we substitute this value for it we shall decrease the value of the benefit which will now be

$$\frac{1}{l_x}\left\{\frac{1}{1\cdot03}\,d_x+\frac{1}{(1\cdot03)^2}\,d_{x+1}+\frac{1}{(1\cdot03)^3}\,d_{x+2}+\,\ldots\right\}$$

$$=\frac{1}{D_x}(C_x+C_{x+1}+C_{x+2}+\,\ldots\,)\text{ at 3 per cent.}$$

$$=A_{x(3\%)}$$

The two approximations given effect to above act in opposite directions, and will to some extent neutralise each other.

9. Under a scheme of Discounted-Bonus or Minimum-Premium Policies the annual premiums are obtained by deducting from the full profit premium the value of a certain rate of bonus.

(*a*). **Cash Bonus.**—At the several investigations cash bonuses are usually declared as a percentage of the premiums received since last investigation. On the assumption that investigations are quinquennial, and that it is desired to apply a cash bonus of $100k$ per cent. of the premiums received in reduction of the full profit premiums, the yearly reduction will be

$$k\times 5P'_x\,\frac{D_{x+5}+D_{x+10}+D_{x+15}+\,\ldots}{N_{x-1}}$$

Now, assuming that $5D_{x+5}=D_{x+3}+D_{x+4}+D_{x+5}+D_{x+6}+D_{x+7}$, etc., we have the yearly reduction equal to

$$kP'_x\,\frac{D_{x+3}+D_{x+4}+\,\ldots\,+D_{x+7}+D_{x+8}+\,\ldots}{N_{x-1}}$$

$$=kP'_x\,\frac{N_{x+2}}{N_{x-1}}$$

$$\text{But again} \quad \frac{5D_{x+5}+5D_{x+10}+5D_{x+15}+\cdots}{D_x}$$

$$= a_x^{(\frac{1}{5})} = a_x - 2$$

$$\text{Therefore} \quad k \times 5P'_x \frac{D_{x+5}+D_{x+10}+D_{x+15}+\cdots}{N_{x-1}}$$

$$= kP'_x \frac{a_x - 2}{a_x + 1}$$

$$= kP'_x\{1 - 3(P_x + d)\}$$

(*b*). **Uniform Reversionary Bonus.**—On the assumption that it is desired to apply the value of a uniform reversionary bonus of $5b$ to be declared every five years, the reduction in the annual with-profit premium will be

$$\frac{5b(M_{x+5}+M_{x+10}+M_{x+15}+\cdots)}{N_{x-1}}$$

If an interim bonus at the same rate is also to be assumed the reduction will be increased to

$$b\frac{R_x}{N_{x-1}}$$

(*c*). **Compound Reversionary Bonus.**—If we apply the value of a similar compound bonus, the yearly reduction will be

$$\frac{5bM_{x+5}+5b(1+5b)M_{x+10}+5b(1+5b)^2M_{x+15}+\cdots}{N_{x-1}}$$

Or including an interim bonus at the same rate, we have

$$\frac{b(R_x - R_{x+5})+b(1+5b)(R_{x+5}-R_{x+10})+b(1+5b)^2(R_{x+10}-R_{x+15})+\cdots}{N_{x-1}}$$

In any such system, if the bonus declared is greater than that applied in reduction of the premium in accordance with any of the above formulas, the excess is added to the sum assured. But if the rate declared be less than the assumed rate, the difference must be deducted from the sum assured, or else an increase must be made in the premium payable, care being taken to ensure that no option is permitted to the disadvantage of the office.

10. To find the annual premium for a pure endowment payable at age $(x+n)$, the premiums received to be returned with simple interest at rate j in the event of previous death, but the premium to be calculated at rate i.

The difficulty here is in the return of the interest on the premiums. In respect of the first premium paid, this return is of the nature of an increasing assurance commencing at one year's interest, and increasing by the same amount per annum. The value of this therefore is

$$j\pi' \frac{R_x - R_{x+n} - nM_{x+n}}{D_x}$$

In respect of the second premium the return is of the nature of a similar assurance deferred one year, and its value is

$$j\pi' \frac{R_{x+1} - R_{x+n} - (n-1)M_{x+n}}{D_x}$$

and so on for each premium, the value of the return of interest in respect of the last being

$$j\pi' \frac{R_{x+n-1} - R_{x+n} - M_{x+n}}{D_x}$$

The value of the return of interest in respect of all the premiums is the summation of these n expressions, and is equal to

$$j\pi' \frac{\Sigma R_x - \Sigma R_{x+n} - nR_{x+n} - \frac{n(n+1)}{2} M_{x+n}}{D_x}$$

Therefore the benefit side

$$= \frac{D_{x+n}}{D_x} + \{\pi(1+\kappa)+c\} \frac{R_x - R_{x+n} - nM_{x+n}}{D_x}$$

$$+ j\{\pi(1+\kappa)+c\} \frac{\Sigma R_x - \Sigma R_{x+n} - nR_{x+n} - \frac{n(n+1)}{2} M_{x+n}}{D_x}$$

Payment side $= \pi \dfrac{N_{x-1} - N_{x+n-1}}{D_x}$

Equating and solving, we have

$$\pi = \Big[D_{x+n} + c(R_x - R_{x+n} - nM_{x+n}) + jc\Big\{\Sigma R_x - \Sigma R_{x+n} - n R_{x+n} - \frac{n(n+1)}{2} M_{x+n}\Big\}\Big]$$
$$\div \Big[N_{x-1} - N_{x+n-1} - (1+\kappa)(R_x - R_{x+n} - nM_{x+n})$$
$$-j(1+\kappa)\Big\{\Sigma R_x - \Sigma R_{x+n} - n R_{x+n} - \frac{n(n+1)}{2} M_{x+n}\Big\}\Big]$$

and $\pi' = \pi(1+\kappa)+c.$

11. To find the annual premium for a similar benefit, with the exception that the premiums are to be returned with *compound* interest at rate j.

Here we have the benefit side

$$= \frac{D_{x+n}}{D_x} + \{\pi(1+\kappa)+c\}\Sigma_1^n v^t_{(i)}(1+j)\frac{(1+j)^t - 1}{j}\frac{d_{x+t-1}}{l_x}$$

$$= \frac{D_{x+n}}{D_x} + \{\pi(1+\kappa)+c\}\frac{1+j}{j}(A'^{\,1}_{x\overline{n|}} - A^{1}_{x\overline{n|}})$$

where $A'^{\,1}_{x\overline{n|}}$ is calculated at rate J, which is such that

$$\frac{1}{1+J} = \frac{1+j}{1+i}$$

Payment side $= \pi \dfrac{N_{x-1} - N_{x+n-1}}{D_x}$

$$\text{Hence } \pi = \frac{D_{x+n} + c\,\dfrac{1+j}{j}\,D_x(A'^{\,1}_{x\overline{n|}} - A^{1}_{x\overline{n|}})}{N_{x-1} - N_{x+n-1} - (1+\kappa)\,\dfrac{1+j}{j}\,D_x(A'^{\,1}_{x\overline{n|}} - A^{1}_{x\overline{n|}})}$$

and $\pi' = \pi(1+\kappa)+c.$

Alternatively, we have the benefit side

$$= \frac{D_{x+n}}{D_x} + \pi'\Big\{(1+j)\frac{M_x - M_{x+n}}{D_x} + (1+j)^2\frac{M_{x+1} - M_{x+n}}{D_x} + \cdots$$
$$+ (1+j)^n \frac{M_{x+n-1} - M_{x+n}}{D_x}\Big\}$$

$$= \frac{D_{x+n}}{D_x} + \{\pi(1+\kappa)+c\}\frac{1}{D_x}\{(1+j)M_x + (1+j)^2 M_{x+1} + \cdots$$
$$+ (1+j)^n M_{x+n-1} - (1+j)s_{\overline{n|}(j)} M_{x+n}\}$$

The payment side being as above, we have

$$\pi = [D_{x+n} + c\{(1+j)M_x + (1+j)^2M_{x+1} + \cdots + (1+j)^nM_{x+n-1} - (1+j)s_{\overline{n|}(j)}M_{x+n}\}] \div [N_{x-1} - N_{x+n-1} - (1+\kappa)\{(1+j)M_x + (1+j)^2M_{x+1} + \cdots + (1+j)^nM_{x+n-1} - (1+j)s_{\overline{n|}(j)}M_{x+n}\}]$$

and $\pi' = \pi(1+\kappa)+c$.

12. The annual premium for a deferred annuity with a similar condition as to return of premiums may be found by substituting N_{x+n} for D_{x+n} in the above formulas.

If in this last problem we assume that the net premiums are returnable, we shall obtain

$$\pi = N_{x+n} \div [N_{x-1} - N_{x+n-1} - \{(1+j)M_x + (1+j)^2M_{x+1} + \cdots + (1+j)^nM_{x+n-1} - (1+j)s_{\overline{n|}(j)}M_{x+n}\}]$$

If, further, we put $j=i$, the portion of the denominator within brackets becomes

$$\begin{aligned}
&(1+i)M_x + (1+i)^2M_{x+1} + \cdots + (1+i)^nM_{x+n-1} - (1+i)s_{\overline{n|}(i)}M_{x+n}\\
&= (1+i)(v^{x+1}d_x + v^{x+2}d_{x+1} + v^{x+3}d_{x+2} + \cdots + v^{x+n+1}d_{x+n} + \cdots)\\
&\qquad + (1+i)^2(v^{x+2}d_{x+1} + v^{x+3}d_{x+2} + \cdots + v^{x+n+1}d_{x+n} + \cdots)\\
&\qquad + \cdots + (1+i)^n(v^{x+n}d_{x+n-1} + v^{x+n+1}d_{x+n} + \cdots)\\
&\quad - \{(1+i) + (1+i)^2 + \cdots + (1+i)^n\}(v^{x+n+1}d_{x+n} + v^{x+n+2}d_{x+n+1} + \cdots)\\
&= v^x(d_x + d_{x+1} + \cdots + d_{x+n-1})\\
&\qquad + v^{x+1}(d_{x+1} + \cdots + d_{x+n-1})\\
&\qquad\quad + \cdots + v^{x+n-1}d_{x+n-1}\\
&= v^x(l_x - l_{x+n}) + v^{x+1}(l_{x+1} - l_{x+n}) + \cdots + v^{x+n-1}(l_{x+n-1} - l_{x+n})\\
&= D_x + D_{x+1} + \cdots + D_{x+n-1} - D_{x+n}\{(1+i)^n + (1+i)^{n-1} + \cdots + (1+i)\}\\
&= N_{x-1} - N_{x+n-1} - D_{x+n}(1+i)s_{\overline{n|}}
\end{aligned}$$

Therefore
$$\begin{aligned}
\pi &= \frac{N_{x+n}}{N_{x-1} - N_{x+n-1} - \{N_{x-1} - N_{x+n-1} - D_{x+n}(1+i)s_{\overline{n|}}\}}\\
&= \frac{N_{x+n}}{D_{x+n}(1+i)s_{\overline{n|}}}\\
&= P_{\overline{n|}}a_{x+n}
\end{aligned}$$

which is the annual premium under a leasehold assurance to provide a_{x+n} at the end of n years certain.

This result is correct; for, since the office has to return to those who die within the n years their contributions along with compound interest at the rate assumed in their calculations, it will derive no benefit from those who so die, and therefore mortality must be left out of account so far as these years are concerned.

13. To find the annual premium limited to t years and returnable with simple interest at rate j for a pure endowment with return.

Here there will be t expressions for the return of interest to sum.

$$\text{Benefit side} = \frac{D_{x+n}}{D_x} + \{\pi(1+\kappa)+c\}\frac{R_x - R_{x+t} - tM_{x+n}}{D_x}$$

$$+j\{\pi(1+\kappa)+c\}\frac{\Sigma R_x - \Sigma R_{x+t} - tR_{x+n} - \dfrac{t(2n-t+1)}{2}M_{x+n}}{D_x}$$

$$\text{Payment side} = \pi\frac{N_{x-1} - N_{x+t-1}}{D_x}$$

π may be found by equating the two sides and solving, and hence also π'.

14. To find the annual premium for a whole-life assurance of 1 deferred n years, premiums to be returnable in the event of death within the n years.

$$\text{Benefit side} = \frac{M_{x+n}}{D_x} + \{\pi(1+\kappa)+c\}\frac{R_x - R_{x+n} - nM_{x+n}}{D_x}$$

$$\text{Payment side} = \pi\frac{N_{x-1}}{D_x}$$

$$\text{Whence} \qquad \pi = \frac{M_{x+n} + c(R_x - R_{x+n} - nM_{x+n})}{N_{x-1} - (1+\kappa)(R_x - R_{x+n} - nM_{x+n})}$$

$$\text{and} \qquad \pi' = \pi(1+\kappa)+c.$$

Mr Stirling gives (*J. I. A.*, xxxi. 259) a simple practical formula for obtaining this annual premium from a table of annual premiums for pure endowments with return.

The argument is as follows:—At the end of the n years the premium for the assurance at the then age would be P_{x+n}, but the office is to continue receiving only the premium π; therefore at that time it must have in hand to meet the shortage in future premiums a sum of

$$(P_{x+n} - \pi)a_{x+n}$$

π must therefore also be the premium for a pure endowment, with return, of this amount, or $(P_{x+n} - \pi)a_{x+n} \times RP_{x\overline{n}|}^{\;\;1}$ where $RP_{x\overline{n}|}^{\;\;1}$ is the annual premium for a pure endowment of 1, with return. That is

$$\pi = (P_{x+n} - \pi)a_{x+n} \times RP_{x\overline{n}|}^{\;\;1}$$

$$\pi\left(\frac{1}{RP_{x\overline{n}|}^{\;\;1}} + a_{x+n}\right) = P_{x+n}a_{x+n}$$

$$\pi = \frac{P_{x+n}a_{x+n}}{\dfrac{1}{RP_{x\overline{n}|}^{\;\;1}} + a_{x+n}}$$

Taking net premiums throughout and substituting for $RP_{x\overline{n}|}^{\;\;1}$ its value as found in *Text Book* formula (31), we get

$$\pi = \frac{M_{x+n}}{N_{x-1} - (R_x - R_{x+n} - nM_{x+n})}$$

which is the annual premium for a deferred assurance with return of the net premium, agreeing with our first formula above if κ and c are zero.

Again, taking the loading as a percentage on the premium only, that is $\pi' = \pi(1+\kappa)$ and $c = 0$, and making the necessary modifications on the value of $RP_{x\overline{n}|}^{\;\;1}$ as found in *Text Book* formula (47), we have by Stirling's formula

$$\pi = \frac{M_{x+n}}{N_{x-1} - (1+\kappa)(R_x - R_{x+n} - nM_{x+n})}$$

which is the annual premium for a deferred assurance with return of the office premium where the office premium is loaded with a percentage on the net premium only, agreeing with our first formula if $c = 0$.

Now, taking $\pi' = \pi(1+\kappa)+c$, and giving to $RP_{\overline{xn}|}^{\;1}$ its value as found in *Text Book* formula (47), we have by Stirling's formula

$$\pi = \frac{A_{x+n}}{\dfrac{N_{x-1}-N_{x+n-1}-(1+\kappa)(R_x-R_{x+n}-nM_{x+n})}{D_{x+n}+c(R_x-R_{x+n}-nM_{x+n})}+a_{x+n}}$$

But by our first formula

$$\pi = \frac{M_{x+n}+c(R_x-R_{x+n}-nM_{x+n})}{N_{x-1}-(1+\kappa)(R_x-R_{x+n}-nM_{x+n})}$$

and these two formulas are not identical.

The reason for the divergency will be found on examining the formula

$$\pi = (P_{x+n}-\pi)a_{x+n} \times RP_{\overline{xn}|}^{\;1}$$

Under the circumstances now being considered $RP_{\overline{xn}|}^{\;1}$ is loaded to provide for the return of $RP_{\overline{xn}|}^{\;1}(1+\kappa)+c$. According to the argument by which this formula was derived the office premium which should therefore be returned is

$$(P_{x+n}-\pi)a_{x+n}\{RP_{\overline{xn}|}^{\;1}(1+\kappa)+c\}$$

But the office premium which actually is to be returned is

$$(P_{x+n}-\pi)a_{x+n}RP_{\overline{xn}|}^{\;1}(1+\kappa)+c$$

and these are obviously not equal.

Mr Stirling, however, put forward the formula merely as a useful method of obtaining the *office* premium for the deferred assurance, the premium P_{x+n} also having to be considered a premium with some loading. Its usefulness is considerable, for the numerator is constant for assurances commencing at age $(x+n)$. The process is to add to a_{x+n} the reciprocal of the office premium for a pure endowment with return, and divide $P_{x+n}a_{x+n}$ by the result.

The formula is easily modified to apply to endowment assurance and limited-payment policies.

For the endowment assurance payable at age $(x+n+m)$, or at death between age $(x+n)$ and that age, with return of premiums if death occurs before age $(x+n)$, we have

$$\pi = \frac{P_{x+n:\overline{m}|}\,a_{x+n:\overline{m}|}}{\dfrac{1}{RP_{\overline{xn}|}^{\;1}}+a_{x+n:\overline{m}|}}$$

For a policy under which premiums are to cease to be payable after age $(x+n+m-1)$, *i.e.* after $(n+m)$ payments, we have

$$\pi = \frac{{}_mP_{x+n}\,a_{x+n:\overline{m}|}}{\dfrac{1}{RP_{\overline{xn}|}^{\,1}} + a_{x+n:\overline{m}|}}$$

15. To find the purchase-price of a life annuity of 1 to (x), subject to the condition that should (x) die before he has received in annuity payments the whole of the purchase-price the balance is to be paid to his estate.

Let W be the purchase-price. Then we have

$$\text{Benefit side} = \frac{N_x}{D_x} + \frac{WM_x - (R_{x+1} - R_{x+W+1})}{D_x}$$

$$\text{Payment side} = W$$

Equating and solving, we have

$$W = \frac{N_x - (R_{x+1} - R_{x+W+1})}{D_x - M_x}$$

Since W is still involved in the right-hand side of the equation it will be necessary to make an approximation to its value in the first place. The right-hand side on being worked out should then agree with the assumed value of W. After two such approximations the true value might be found by interpolation.

This method of obtaining W is not quite correct, inasmuch as W is usually an integer plus a fraction. But as Mr Manly, the author of the formula, points out, the correction for the value of the assurance of this fraction of the annuity is so insignificant that it might be ignored.

These remarks apply also to the two following problems:—

To find the single premium for an annuity with a similar condition but deferred n years, the net premium also being returnable in the event of death within n years.

Let W be the purchase-price of the annuity.

$$\text{Benefit side} = \frac{N_{x+n}}{D_x} + \frac{WM_x - (R_{x+n+1} - R_{x+n+W+1})}{D_x}$$

$$\text{Payment side} = W$$

$$\text{Hence we have } W = \frac{N_{x+n} - (R_{x+n+1} - R_{x+n+W+1})}{D_x - M_x}$$

To find the annual premium for a similar annuity to the last, with the condition that all net premiums paid are to be returnable in the event of death within the n years.

$$\text{Benefit side} = \frac{N_{x+n}}{D_x} + \frac{\pi(R_x - R_{x+n}) - (R_{x+n+1} - R_{x+n+n\pi+1})}{D_x}$$

$$\text{Payment side} = \pi\frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$\text{Hence } \pi = \frac{N_{x+n} - (R_{x+n+1} - R_{x+n+n\pi+1})}{N_{x-1} - N_{x+n-1} - (R_x - R_{x+n})}$$

16. To find the annual premium for a pure endowment payable at age $(x+n)$; the premiums to be payable only so long as another life aged y is alive jointly with (x), and to be returnable if (x) should die within the n years.

The value of the return in question was discussed on page 129, and making use of the result there given we have

Benefit side

$$= \frac{D_{x+n}}{D_x} + \{\pi(1+\kappa)+c\}\left(\frac{M_x - M_{x+n}}{D_x} + \frac{M_{x+1} - M_{x+n}}{D_x}p_y + \cdots + \frac{M_{x+n-1} - M_{x+n}}{D_x}{}_{n-1}p_y\right)$$

$$\text{Payment side} = \pi\frac{N_{x-1:y-1} - N_{x+n-1:y+n-1}}{D_{xy}}$$

Hence we may obtain π. Also $\pi' = \pi(1+\kappa)+c$.

17. To find the annual premium for a similar benefit, but the return of premiums to be with simple interest at rate j.

Following the method of the solution on page 308, we have

Benefit side

$$= \frac{D_{x+n}}{D_x} + \{\pi(1+\kappa)+c\}\left(\frac{M_x - M_{x+n}}{D_x} + \frac{M_{x+1} - M_{x+n}}{D_x}p_y + \cdots + \frac{M_{x+n-1} - M_{x+n}}{D_x}{}_{n-1}p_y\right)$$

$$+j\{\pi(1+\kappa)+c\}\left(\frac{R_x - R_{x+n} - nM_{x+n}}{D_x} + \frac{R_{x+1} - R_{x+n} - \overline{n-1}M_{x+n}}{D_x}p_y + \cdots + \frac{R_{x+n-1} - R_{x+n} - M_{x+n}}{D_x}{}_{n-1}p_y\right)$$

$$\text{Payment side} = \pi\frac{N_{x-1:y-1}-N_{x+n-1:y+n-1}}{D_{xy}}$$

Equating and solving, π, and hence π', may be found.

18. To find the annual premium for a similar benefit, but the premiums to be returnable with compound interest at rate j in the event of (x)'s death within n years.

Benefit side

$$= \frac{D_{x+n}}{D_x}+\{\pi(1+\kappa)+c\}\Big[\Big\{(1+j)\frac{C_x}{D_x}+(1+j)^2\frac{C_{x+1}}{D_x}+\cdots$$
$$+(1+j)^n\frac{C_{x+n-1}}{D_x}\Big\}$$
$$+\Big\{(1+j)\frac{C_{x+1}}{D_x}+(1+j)^2\frac{C_{x+2}}{D_x}+\cdots+(1+j)^{n-1}\frac{C_{x+n-1}}{D_x}\Big\}p_y+\cdots$$
$$+\Big\{(1+j)\frac{C_{x+n-1}}{D_x}\Big\}{}_{n-1}p_y\Big]$$
$$= \frac{D_{x+n}}{D_x}+\{\pi(1+\kappa)+c\}\Big\{\frac{M'_x-M'_{x+n}}{D'_x}+\frac{1}{1+j}\frac{M'_{x+1}-M'_{x+n}}{D'_x}p_y$$
$$+\cdots+\frac{1}{(1+j)^{n-1}}\frac{M'_{x+n-1}-M'_{x+n}}{D'_x}{}_{n-1}p_y\Big\}$$

where D'_x, M'_x, etc., are calculated at rate J, which is such that

$$\frac{1}{1+J} = \frac{1+j}{1+i}$$

$$\text{Payment side} = \pi\frac{N_{x-1:y-1}-N_{x+n-1:y+n-1}}{D_{xy}}$$

whence π and π' may be found.

19. To find the annual premium for an assurance on the life of (x) deferred n years; the premium during that period to be payable only if (y) also is alive, and thereafter throughout (x)'s life, and to be returnable should (x) die within the n years.

Benefit side

$$= \frac{M_{x+n}}{D_x}+\{\pi(1+\kappa)+c\}\Big(\frac{M_x-M_{x+n}}{D_x}+\frac{M_{x+1}-M_{x+n}}{D_x}p_y+\cdots$$
$$+\frac{M_{x+n-1}-M_{x+n}}{D_x}{}_{n-1}p_y\Big)$$

Payment side $= \pi\left(\frac{N_{x-1:y-1} - N_{x+n-1:y+n-1}}{D_{xy}} + \frac{N_{x+n-1}}{D_x}\right)$

Equating these two sides we may solve for π, and $\pi' = \pi(1+\kappa)+c$.

20. To find the single premium for an annuity to the last survivor of (x) and (y) deferred n years, the premium to be returned in the event of the annuity not being entered upon.

Let a' be the purchase-price. Then we have

$$\begin{aligned}\text{Benefit side} &= {}_{n|}a_{\overline{xy}} + a' \times {}_{|n}A_{\overline{xy}} \\ &= ({}_{n|}a_x + {}_{n|}a_y - {}_{n|}a_{xy}) \\ &\qquad + \{a(1+\kappa)+c\}({}_{|n}A_x + {}_{|n}A_y - {}_{|n}A_{xy})\end{aligned}$$

Payment side $= a$.

Hence equating and solving for a, we have

$$a = \frac{({}_{n|}a_x + {}_{n|}a_y - {}_{n|}a_{xy}) + c({}_{|n}A_x + {}_{|n}A_y - {}_{|n}A_{xy})}{1-(1+\kappa)({}_{|n}A_x + {}_{|n}A_y - {}_{|n}A_{xy})}$$

and $a' = a(1+\kappa)+c$.

To find the annual premium for a similar benefit, all premiums paid to be returned on the death of the survivor should that occur within n years.

It has already been pointed out (page 151) that some difficulty attends the fixing of the status during which the premium shall be payable. We may consider both cases.

(a) If the joint lives be taken as the status, the benefit side

$$\begin{aligned}= {}_{n|}a_{\overline{xy}} + \{\pi(1+\kappa)+c\}\Bigg\{&\left(\frac{M_x - M_{x+n}}{D_x} + \frac{M_{x+1} - M_{x+n}}{D_x}p_y + \cdots\right. \\ &\left.+ \frac{M_{x+n-1} - M_{x+n}}{D_x}{}_{n-1}p_y\right) + \left(\frac{M_y - M_{y+n}}{D_y} + \frac{M_{y+1} - M_{y+n}}{D_y}p_x + \cdots\right. \\ &\left.+ \frac{M_{y+n-1} - M_{y+n}}{D_y}{}_{n-1}p_x\right) - \frac{R_{xy} - R_{x+n:y+n} - nM_{x+n:y+n}}{D_{xy}}\Bigg\}\end{aligned}$$

Payment side $= \pi(1 + a_{xy:\overline{n-1}|})$

whence we may obtain π and also π'.

(*b*) If the premium be payable to the death of the last survivor, we have

$$\text{Benefit side} = {}_{n|}a_{\overline{xy}} + \{\pi(1+\kappa)+c\}\left(\frac{R_x - R_{x+n} - nM_{x+n}}{D_x}\right.$$

$$\left. + \frac{R_y - R_{y+n} - nM_{y+n}}{D_y} - \frac{R_{xy} - R_{x+n:y+n} - nM_{x+n:y+n}}{D_{xy}}\right)$$

$$\text{Payment side} = \pi(1 + a_{\overline{xy}:\overline{n-1|}})$$

from which other values of π and π' will be found.

21. In the problems connected with pure endowments with return of premiums, the element of mortality is in practice frequently ignored. This is in effect taking for granted that the life will survive the term; and if it does not, the office receives for its trouble interest on the premiums which it has received and now has to repay.

Thus in the case of the annual premium for a child's pure endowment to (x) payable at the end of n years, with return of premiums in the event of previous death, the net premium is simply found from

$$\pi = \frac{v^n}{\mathbf{a}_{\overline{n|}}}$$

When the question is complicated by making the premium payable only so long as the father (y) shall survive (see page 315), the net premium is taken as

$$\pi = \frac{v^n}{\mathbf{a}_{y\overline{n|}}}$$

EXAMPLES

1. The sum of £s is deposited by each of l_x persons in a fund, and accumulated at compound interest. £a is paid on the death of each member, at the end of the year in which he dies, and at the end of n years the amount remaining in the fund is applied to the purchase of an annuity upon the life of each of the surviving members. Find the amount of the annuity.

Let the amount of the annuity be p. Then the value of an annuity of p to each of the survivors of l_x persons alive n years ago is $pl_{x+n}a_{x+n}$.

But the accumulations of the fund are

$$sl_x(1+i)^n - a\{d_x(1+i)^{n-1} + d_{x+1}(1+i)^{n-2} + \dots + d_{x+n-1}\}$$

Hence

$$p = \frac{sl_x(1+i)^n - a\{d_x(1+i)^{n-1} + d_{x+1}(1+i)^{n-2} + \dots + d_{x+n-1}\}}{l_{x+n}a_{x+n}}$$

$$= \frac{sD_x - a(M_x - M_{x+n})}{N_{x+n}}$$

Alternatively, p being the amount of the annuity as before,

$$\text{Benefit side} = \frac{a(M_x - M_{x+n})}{D_x} + \frac{pN_{x+n}}{D_x}$$

$$\text{Payment side} = s$$

$$\text{Equating, we have } \frac{a(M_x - M_{x+n})}{D_x} + \frac{pN_{x+n}}{D_x} = s$$

$$\frac{pN_{x+n}}{D_x} = \frac{sD_x - a(M_x - M_{x+n})}{D_x}$$

$$p = \frac{sD_x - a(M_x - M_{x+n})}{N_{x+n}} \text{ as before.}$$

2. If l_x persons each secure by annual premium an endowment, show that the amount which will be payable at maturity to the survivors consists of the accumulated premiums paid by the survivors and by those who die.

The annual premium is $P_{x\overline{n}|}^{\ \,1} = \dfrac{D_{x+n}}{N_{x-1} - N_{x+n-1}}$

and its accumulations to the end of n years amount to

$$P_{x\overline{n}|}^{\ \,1}\{l_x(1+i)^n + l_{x+1}(1+i)^{n-1} + \dots + l_{x+n-1}(1+i)\}$$

$$= P_{x\overline{n}|}^{\ \,1}\left(\frac{v^x l_x + v^{x+1}l_{x+1} + \dots + v^{x+n-1}l_{x+n-1}}{v^{x+n}}\right)$$

$$= P_{x\overline{n}|}^{\ \,1}\,\frac{N_{x-1} - N_{x+n-1}}{v^{x+n}}$$

$$= \frac{D_{x+n}}{N_{x-1} - N_{x+n-1}} \times \frac{N_{x-1} - N_{x+n-1}}{v^{x+n}}$$

$$= l_{x+n}$$

which is the amount payable at maturity, being 1 for each of the l_{x+n} survivors.

3. For what benefit is $\frac{P_{\overline{x\overline{n}|}}}{d}$ the single premium? Explain the formula verbally.

This is the single premium for an endowment assurance of 1 with the net premium returnable, since the value of such a benefit is

$$A_{\overline{x\overline{n}|}} + A \times A_{\overline{x\overline{n}|}}$$

the payment for it being A.

$$\text{Hence } A = \frac{A_{\overline{x\overline{n}|}}}{1 - A_{\overline{x\overline{n}|}}}$$

$$= \frac{P_{\overline{x\overline{n}|}}}{d}$$

$$\text{for } P_{\overline{x\overline{n}|}} = \frac{dA_{\overline{x\overline{n}|}}}{1 - A_{\overline{x\overline{n}|}}}$$

Now $\frac{P_{\overline{x\overline{n}|}}}{d}$ is the value of a perpetuity-due of $P_{\overline{x\overline{n}|}}$. But $P_{\overline{x\overline{n}|}}$ will insure the payment of 1 at the failure of the joint status of (x) and $\overline{n}|$; and after that, a fresh status of a similar kind being set up and the payments of premium continuing under the perpetuity, payment of 1 will be made on the failure of the second status; and so on indefinitely. And this is the benefit asked for, since on payment of the endowment assurance 1 may be taken, and there remains A to set up a second similar contract, and so on indefinitely.

4. "Suppose the annual premiums to increase or decrease a certain sum every t years, and at the end of v intervals of t years each the premium to continue constant during the remainder of life, what annual premium should be required during the first t years"? Jones gives as the answer to this question

$$P = \frac{M_x \mp q(N_{x+t-1} + N_{x+2t-1} + \cdot \cdot \cdot + N_{x+vt-1})}{N_{x-1}}$$

while Chisholm, correcting him, gives

$$P = \frac{M_x}{N_{x-1} \pm q(N_{x+t-1} + N_{x+2t-1} + \cdot \cdot \cdot + N_{x+vt-1})}$$

State the different conditions under which both answers are correct.

This problem is discussed in *Text Book*, Articles 28, 29, and 35 of this chapter, and from what is shown there it will be observed that Jones's solution proceeds on the assumption that the premium increases or decreases by q per unit of the sum assured, while Chisholm assumes the increase or decrease to be q per unit of the premium. Each answer in its own case is correct, the question being stated ambiguously.

5. Find the annual premium for an annuity to (x) after death of (y), all premiums paid except the first to be returned in the event of (x) dying before (y).

$$\text{Benefit side} = a_{y|x} + \{\pi(1+\kappa)+c\}\frac{R_{\overset{1}{x+1}:\overline{y+1}}}{D_{xy}}$$

$$\text{Payment side} = \pi(1+a_{xy})$$

$$\text{And } \pi = \frac{a_x - a_{xy} + c\,\dfrac{R_{\overset{1}{x+1}:\overline{y+1}}}{D_{xy}}}{1 + a_{xy} - (1+\kappa)\,\dfrac{R_{\overset{1}{x+1}:\overline{y+1}}}{D_{xy}}}$$

6. Deduce a formula for the annual premium for an assurance on the life of (x) against (y) for n years, with return of all premiums paid should (y) predecease (x).

Benefit side

$$= \frac{M^1_{xy} - M_{\overset{1}{x+n}:\overline{y+n}}}{D_{xy}} + \{\pi(1+\kappa)+c\}\frac{R_{x\overset{1}{y}} - R_{\overline{x+n}:\overset{1}{y+n}} - nM_{\overline{x+n}:\overset{1}{y+n}}}{D_{xy}}$$

$$\text{Payment side} = \pi\,\frac{N_{x-1:y-1} - N_{x+n-1:y+n-1}}{D_{xy}}$$

whence

$$\pi = \frac{(M^1_{xy} - M_{\overset{1}{x+n}:\overline{y+n}}) + c(R_{x\overset{1}{y}} - R_{\overline{x+n}:\overset{1}{y+n}} - nM_{\overline{x+n}:\overset{1}{y+n}})}{(N_{x-1:y-1} - N_{x+n-1:y+n-1}) - (1+\kappa)(R_{x\overset{1}{y}} - R_{\overline{x+n}:\overset{1}{y+n}} - nM_{\overline{x+n}:\overset{1}{y+n}})}$$

7. Find the annual premium required to secure to (x) a pure endowment of 1 payable at the end of 20 years, with return of two-thirds of the premiums in the event of death within the second half of the period.

Benefit side

$$= \frac{D_{x+20}}{D_x} + \tfrac{2}{3}\{\pi(1+\kappa)+c\}\frac{10M_{x+10}+R_{x+10}-R_{x+20}-20M_{x+20}}{D_x}$$

$$\text{Payment side} = \pi\frac{N_{x-1}-N_{x+19}}{D_x}$$

whence

$$\pi = \frac{D_{x+20}+\tfrac{2}{3}c(10M_{x+10}+R_{x+10}-R_{x+20}-20M_{x+20})}{N_{x-1}-N_{x+19}-\tfrac{2}{3}(1+\kappa)(10M_{x+10}+R_{x+10}-R_{x+20}-20M_{x+20})}$$

8. Find the annual premium limited to t payments for a whole-life assurance to (x) subject to the condition that interest for each year on the net premiums, up to and including the year of death, is to be allowed by the office at rate i, which is the rate realised by the office on its investments.

$$\text{Benefit side} = \frac{M_x}{D_x} + i\pi\frac{(S_x-S_{x+t})+(R_x-R_{x+t})}{D_x}$$

$$\text{Payment side} = \pi\frac{N_{x-1}-N_{x+t-1}}{D_x}\ .$$

$$\text{whence}\quad \pi = \frac{M_x}{(N_{x-1}-iS_x-iR_x)-(N_{x+t-1}-iS_{x+t}-iR_{x+t})}$$

$$= \frac{M_x}{R_x-R_{x+t}}$$

$$\begin{aligned}\text{since}\quad N_{x-1}-iS_x-iR_x &= S_{x-1}-S_x-iS_x-viS_{x-1}+iS_x\\ &= vS_{x-1}-S_x\\ &= R_x\end{aligned}$$

$$\text{and similarly}\quad N_{x+t-1}-iS_{x+t}-iR_{x+t} = R_{x+t}$$

A proof of this formula by general reasoning similar to that of *Text Book,* Article 66, may be given.

The office can gain nothing from accumulation of interest on the premiums, and therefore the payment side is the value of a benefit of π payable at the end of the first year if death occur in the first year, 2π payable at the end of the second year if death occur in the second year, and so on, increasing up to the tth year, after which the increase ceases, and the benefit remains at $t\pi$ till

the year of death. The benefit side is simply the present value of 1 payable at the end of the year of death.

9. Find the annual premium for an endowment assurance to (x) payable at age $(x+t)$ or previous death, subject to the condition that interest for each year on the net premiums, up to and including the tth year or the year of previous death, is to be allowed by the office at rate i, which is the rate realised by the office on its investments.

Benefit side

$$= \frac{M_x - M_{x+t} + D_{x+t}}{D_x} + i\pi\left\{\frac{S_x - S_{x+t} - tN_{x+t}}{D_x} + \frac{R_x - R_{x+t} - tM_{x+t}}{D_x}\right\}$$

$$\text{Payment side} = \pi\frac{N_{x-1} - N_{x+t-1}}{D_x}$$

whence

$$\pi = \frac{M_x - M_{x+t} + D_{x+t}}{(N_{x-1} - iS_x - iR_x) - (N_{x+t-1} - iS_{x+t} - iR_{x+t}) + ti(N_{x+t} + M_{x+t})}$$

$$= \frac{M_x - M_{x+t} + D_{x+t}}{R_x - R_{x+t} - tM_{x+t} + tD_{x+t}}$$

since as before $N_{x-1} - iS_x - iR_x = R_x$

and $N_{x+t-1} - iS_{x+t} - iR_{x+t} = R_{x+t}$

and since

$$\begin{aligned} i(N_{x+t} + M_{x+t}) &= iN_{x+t} + viN_{x+t-1} - iN_{x+t} \\ &= (1-v)N_{x+t-1} \\ &= D_{x+t} + N_{x+t} - vN_{x+t-1} \\ &= D_{x+t} - M_{x+t} \end{aligned}$$

A proof of this by general reasoning may also be given. The value of the payment side is as before that of an increasing assurance of π, 2π, etc., up to $t\pi$ in the tth year. But in this case the benefit ceases then entirely, $t\pi$ being receivable also if the life completes that year. The benefit side is simply the present value of 1, payable at the end of t years, or at the end of the year of previous death.

10. Find the annual premium for a pure endowment payable at age $(x+t)$; the premiums to be limited to n payments, and to be returnable in the event of death before age $(x+t)$.

$$\text{Benefit side} = \frac{D_{x+t}}{D_x} + \{\pi(1+\kappa)+c\}\frac{R_x - R_{x+n} - nM_{x+t}}{D_x}$$

$$\text{Payment side} = \pi\,\frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$\pi = \frac{D_{x+t} + c(R_x - R_{x+n} - nM_{x+t})}{(N_{x-1} - N_{x+n-1}) - (1+\kappa)(R_x - R_{x+n} - nM_{x+t})}$$

11. Obtain a formula for the office annual premium, P, required for a policy on (x) for a term of n years, the assurance to cover (1) an advance of £p made out of a trust fund at the beginning of each year, (2) the premiums actually paid under the policy, and (3) the legal expenses of the arrangement, say £a.

Benefit side

$$= p\,\frac{R_x - R_{x+n} - nM_{x+n}}{D_x} + \{\pi(1+\kappa)+c\}\frac{R_x - R_{x+n} - nM_{x+n}}{D_x}$$
$$+ a\,\frac{M_x - M_{x+n}}{D_x}$$

$$\text{Payment side} = \pi\,\frac{N_{x-1} - N_{x+n-1}}{D_x}$$

$$\text{whence } \pi = \frac{(p+c)(R_x - R_{x+n} - nM_{x+n}) + a(M_x - M_{x+n})}{(N_{x-1} - N_{x+n-1}) - (1+\kappa)(R_x - R_{x+n} - nM_{x+n})}$$

and $P = \pi(1+\kappa)+c$.

12. Obtain, in terms of commutation symbols and the rate of interest, an expression for the annual premium for a deferred annuity to be entered on at age 65 on a life now aged 40, the premium to be returnable in case of death before 65, and the annuity to be payable half-yearly, and to be complete.

$$\text{Benefit side} = \frac{1}{D_{40}}\Big[N_{65} + \tfrac{1}{4}D_{65} + \tfrac{1}{4}(1+i)^{\frac{1}{2}}M_{65}$$
$$+ \{P(1+\kappa)+c\}(R_{40} - R_{65} - 25M_{65})\Big]$$

$$\text{Payment side} = P\,\frac{N_{39} - N_{64}}{D_{40}}$$

whence $P = \dfrac{N_{65}+\frac{1}{4}D_{65}+\frac{1}{4}(1+i)^{\frac{1}{2}}M_{65}+c(R_{40}-R_{65}-25M_{65})}{N_{39}-N_{64}-(1+\kappa)(R_{40}-R_{65}-25M_{65})}$

and $P' = P(1+\kappa)+c.$

13. Required the net single premium for an insurance upon the life of (x) of £1000, increasing at compound interest during the first 5 years at the rate of 4 per cent. per annum.

$$\text{The single premium} = \frac{1000}{D_x}\{(1{\cdot}04)C_x+(1{\cdot}04)^2C_{x+1}+(1{\cdot}04)^3C_{x+2}$$
$$+(1{\cdot}04)^4C_{x+3}+(1{\cdot}04)^5M_{x+4}\}$$

14. An office proposes to grant endowment assurances under a table of premiums reduced by anticipation of future bonus. Investigate a formula for the annual reduction when the discounted bonus is

(*a*) A quinquennial cash bonus of 1 per cent. per annum of the premium.

(*b*) A simple reversionary bonus of £1 per cent. per annum declared quinquennially and vesting when declared. It may be assumed that the office does not allow interim bonuses.

(*a*) For a cash bonus the reduction is

$$\cdot 01\,P'_{\overline{xn}|}\frac{5(D_{x+5}+D_{x+10}+D_{x+15}+\cdots+D_{x+n})}{N_{x-1}-N_{x+n-1}}$$

(*b*) And for a simple reversionary bonus

$$\cdot 01\times\frac{5(M_{x+5}+M_{x+10}+M_{x+15}+\cdots+M_{x+n-5})-(n-5)M_{x+n}+nD_{x+n}}{N_{x-1}-N_{x+n-1}}$$

15. Obtain a formula for the office annual premium for an endowment assurance policy on (x) to mature in 20 years, the premium to be based on select tables, and to provide for a compound reversionary bonus of p per cent. per annum declared quinquennially, with interim bonuses at the same rate after the first 5 years, and the loading to provide for an initial commission of k per cent. on the sum assured spread over the whole term, a constant of l per cent. on the sum assured, and a percentage of m on the gross premium.

The problem to find the net premium for such an assurance has been discussed in general terms on page 305. Here the particular case is treated, and the office premium also deduced.

First, to find the net premium, we have benefit side

$$
\begin{aligned}
= \frac{1}{D_{[x]}}\Big\{&(M_{[x]} - M_{[x]+5}) + \left(1+\frac{5p}{100}\right)(M_{[x]+5} - M_{[x]+10}) \\
&+ \left(1+\frac{5p}{100}\right)^2(M_{[x]+10} - M_{[x]+15}) \\
&+ \left(1+\frac{5p}{100}\right)^3(M_{[x]+15} - M_{[x]+20}) + \left(1+\frac{5p}{100}\right)^4 D_{[x]+20} \\
&+ \frac{p}{100}\left(1+\frac{5p}{100}\right)(R_{[x]+5} - R_{[x]+10} - 5M_{[x]+10}) \\
&+ \frac{p}{100}\left(1+\frac{5p}{100}\right)^2(R_{[x]+10} - R_{[x]+15} - 5M_{[x]+15}) \\
&+ \frac{p}{100}\left(1+\frac{5p}{100}\right)^3(R_{[x]+15} - R_{[x]+20} - 5M_{[x]+20})\Big\}
\end{aligned}
$$

$$
\text{Payment side} = \pi\,\frac{N_{[x]} - N_{[x]+20}}{D_{[x]}}
$$

whence

$$
\begin{aligned}
\pi = \frac{1}{N_{[x]} - N_{[x]+20}}\Big\{&M_{[x]} + \frac{5p}{100}M_{[x]+5} + \frac{p}{100}\left(1+\frac{5p}{100}\right)(R_{[x]+5} - R_{[x]+10}) \\
&+ \frac{p}{100}\left(1+\frac{5p}{100}\right)^2(R_{[x]+10} - R_{[x]+15}) + \frac{p}{100}\left(1+\frac{5p}{100}\right)^3(R_{[x]+15} - R_{[x]+20}) \\
&+ \left(1+\frac{5p}{100}\right)^4(D_{[x]+20} - M_{[x]+20})\Big\}
\end{aligned}
$$

Further, the office premium

$$
\pi' = \frac{1}{1-\dfrac{m}{100}}\left(\pi + \frac{k}{100a_{[x]:\overline{20}|}} + \frac{l}{100}\right)
$$

16. Find the annual premium for a deferred annuity-due to (x), the first payment of which is to be made at age $(x+n)$ with premiums returnable in the event of death before that age. If (x) survive the n years, the annuity is guaranteed for at least t years whether he live so long or not.

$$\text{Benefit side} = \frac{D_{x+n}}{D_x}(a_{\overline{t}|} + {}_t|a_{x+n}) + \{\pi(1+\kappa)+c\}\frac{R_x - R_{x+n} - nM_{x+n}}{D_x}$$

$$\text{Payment side} = \pi a_{\overline{xn}|}$$

$$\text{Equating, } \pi = \frac{D_{x+n}(a_{\overline{t}|} + {}_t|a_{x+n}) + c(R_x - R_{x+n} - nM_{x+n})}{(N_{x-1} - N_{x+n-1}) - (1+\kappa)(R_x - R_{x+n} - nM_{x+n})}$$

$$\text{and} \qquad \pi' = \pi(1+\kappa)+c.$$

17. Find by the $O^{[M]}$ table at $3\frac{1}{2}$ per cent. the annual premium at age at entry 30 for an endowment assurance of 1 for a term of 30 years with a uniform reversionary bonus of 1 per cent. on the sum assured in respect of each year entered upon after the first.

Benefit side

$$= \frac{M_{[30]} - M_{60} + D_{60}}{D_{[30]}} + \cdot 01\,\frac{R_{[30]+1} - R_{60} - 29M_{60} + 29D_{60}}{D_{[30]}}$$

$$\text{Payment side} = P\,\frac{N_{[30]} - N_{60}}{D_{[30]}}$$

$$\text{whence } P = \frac{M_{[30]} - M_{60} + D_{60} + \cdot 01\,(R_{[30]+1} - R_{60} - 29M_{60} + 29D_{60})}{N_{[30]} - N_{60}}$$

$$= \frac{(10267\cdot 77 - 4671\cdot 51 + 7432\cdot 6) + \cdot 01(278665\cdot 44 - 58079\cdot 89 - 29 \times 4671\cdot 51 + 29 \times 7432\cdot 6)}{614584 - 81650\cdot 3}$$

$$= \frac{13028\cdot 86 + 3006\cdot 5716}{532933\cdot 7}$$

$$= \frac{16035\cdot 4316}{532933\cdot 7}$$

$$= \cdot 03009$$

18. Find by the $O^{[M]}$ table at $3\frac{1}{2}$ per cent., for age at entry 40 with a term of 15 years, the annual premium for an endowment assurance of 1, together with a compound reversionary bonus of 1 per cent. per annum, which is to compound every 5 years, with an interim bonus at the same rate in the event of death during the 5 years calculated on the sum assured and bonuses in force at the beginning of the quinquennium.

$$\text{Benefit side} = \frac{1}{D_{[40]}}\Big[(M_{[40]} - M_{55} + D_{55}) + \cdot 01(R_{[40]} - R_{[40]+5})$$
$$+ \cdot 01(1\cdot 05)(R_{[40]+5} - R_{50}) + \cdot 01(1\cdot 05)^2(R_{50} - R_{55})$$
$$- M_{55}\{(1\cdot 05)^3 - 1\} + D_{55}\{(1\cdot 05)^3 - 1\}\Big]$$

$$\text{Payment side} = P\,\frac{N_{[40]} - N_{55}}{D_{[40]}}$$

whence

$$P = \frac{1}{N_{[40]} - N_{55}}\Big[(M_{[40]} - M_{55} + D_{55}) + \cdot 01(R_{[40]} - R_{[40]+5})$$
$$+ \cdot 01(1\cdot 05)(R_{[40]+5} - R_{50}) + \cdot 01(1\cdot 05)^2(R_{50} - R_{55})$$
$$+ \{(1\cdot 05)^3 - 1\}(D_{55} - M_{55})\Big]$$

$$= \frac{1}{352204 - 126234\cdot 5}\Big\{(8202\cdot 99 - 5689\cdot 39 + 9958\cdot 2)$$
$$+ \cdot 01(191831\cdot 83 - 151953\cdot 13)$$
$$+ \cdot 01(1\cdot 05)(151953\cdot 13 - 115901\cdot 12)$$
$$+ \cdot 01(1\cdot 1025)(115901\cdot 12 - 84508\cdot 96)$$
$$+ \cdot 157625(9958\cdot 2 - 5689\cdot 39)\Big\}$$

$$= \frac{14268\cdot 1029}{225969\cdot 5}$$

$$= \cdot 06314$$

19. Given that the office single premiums for pure endowments of £100 at a certain age, with and without return of premium in event of previous death, are £70 and £65 respectively, find the office single premium for the pure endowment with return of half the premium in event of previous death.

As pointed out on page 299, it would be wrong to charge £67, 10s., as that would suffice for the return of £35 (*i.e.*, half of £70) and not of £33, 15s. in event of previous death.

Instead, let S be the sum assured under a policy without return, and (100 − S) the sum assured under a policy with return, so that the premiums on the two policies shall be equal. That is

$$(100 - S) \times \cdot 7 = S \times \cdot 65$$
$$S = 51\cdot 852$$
$$\text{and } 100 - S = 48\cdot 148$$

The premium under each policy will be 33·704, and the premium required under the joint policy will therefore be £67, 8s. 2d.

20. Find, by the use of the following office rates, viz.:—

$$P'_{40:\overline{10}|} = \cdot 09479,\ P'^{\,1}_{40:\overline{10}|} = \cdot 01475, \text{ and } A'_{50} = \cdot 55396,$$

the annual premiums for an assurance on a male life, aged 40 next birthday, of £1000, payable in the event of death within 10 years, with a return of all the premiums paid if he survive that term :—

(*a*) If the return of premium is to be made at the end of 10 years, and (*b*) if it is not to be made till death.

Let P be the premium required.

Then (*a*) $P = 1000\,P'^{\,1}_{40:\overline{10}|} + 10\,P \times P'_{40:\overline{10}|}{}^{\!\!1}$

$$\therefore P = \frac{1000\,P'^{\,1}_{40:\overline{10}|}}{1 - 10\,P'_{40:\overline{10}|}{}^{\!\!1}}$$

Now $P'_{40:\overline{10}|}{}^{\!\!1}$ is not given, but may be taken

$$= P'_{40:\overline{10}|} - P'^{\,1}_{40:\overline{10}|} = \cdot 09479 - \cdot 01475 = \cdot 08004.$$

Though by this method there is insufficient loading on the pure endowment portion of the premium, yet the fact that there is a term assurance for a very much larger figure makes up for this loss.

$$\text{Therefore } P = \frac{1000 \times \cdot 01475}{1 - 10 \times \cdot 08004}$$

$$= \frac{14 \cdot 75}{\cdot 1996}$$

$$= 73 \cdot 898$$

$$\text{Again } (b)\ P = 1000\,P'^{\,1}_{40:\overline{10}|} + 10\,P \times P'_{40:\overline{10}|}{}^{\!\!1} A'_{50}$$

$$P = \frac{1000\,P'^{\,1}_{40:\overline{10}|}}{1 - 10\,P'_{40:\overline{10}|}{}^{\!\!1} A'_{50}}$$

$$= \frac{14 \cdot 75}{1 - \cdot 8004 \times \cdot 55396}$$

$$= \frac{14 \cdot 75}{\cdot 55661}$$

$$= 26 \cdot 5$$

If in these formulas we assume net premiums throughout, we get $(a) = \dfrac{1000\,A^{\;1}_{40:\overline{10}|}}{a_{40:\overline{10}|} - 10A^{\;1}_{40:\overline{10}|}}$, and $(b) = \dfrac{1000\,A^{\;1}_{40:\overline{10}|}}{a_{40:\overline{10}|} - 10A^{\;2}_{40:\overline{10}|}}$, which are obviously the net annual premiums for the benefits, thus proving the correctness of our formulas.

21. Under a certain "Reversible Premium" scheme, an office undertakes to grant whole-life assurances, under which, after n years, the company pays the premiums to the assured, instead of receiving them from him, and this continues until the life drops, when the sum assured becomes payable. If $n = 20$, find the office annual premium at age 40 by the *Text Book* table at 3 per cent., allowing for a loading of 15 per cent. of the net premium, and 5s. per cent. on the sum assured.

$$\text{Benefit side} = \frac{M_x}{D_x} + \frac{\pi' N_{x+n-1}}{D_x}$$

$$\text{Payment side} = \frac{\pi(N_{x-1} - N_{x+n-1})}{D_x}$$

$$\text{whence } \pi = \frac{M_x + cN_{x+n-1}}{N_{x-1} - (2+\kappa)N_{x+n-1}}$$

Under the given conditions

$$\begin{aligned}
\pi &= \frac{M_{40} + \cdot0025N_{59}}{N_{39} - 2\cdot15N_{59}} \\
&= \frac{11869\cdot4 + \cdot0025 \times 112093\cdot8}{458461 - 2\cdot15 \times 112093\cdot8} \\
&= \frac{12149\cdot6345}{217459\cdot33} \\
&= \cdot05587 \\
\text{and } \pi' &= \cdot05587 \times 1\cdot15 + \cdot0025 \\
&= \cdot06675 \\
&= \text{£6, 13s. 6d. per cent.}
\end{aligned}$$

22. Find the annual premium for a pure endowment payable at age $(x+n)$; the premiums to be limited to t, and to be returned with compound interest at rate j in the event of death before age $(x+n)$.

Benefit side

$$= \frac{D_{x+n}}{D_x} + \{\pi(1+\kappa)+c\}\left\{(1+j)\frac{M_x - M_{x+t}}{D_x} + (1+j)^2\frac{M_{x+1} - M_{x+t}}{D_x}\right.$$

$$+ \cdots + (1+j)^t \frac{M_{x+t-1} - M_{x+t}}{D_x}$$

$$\left. + (1+j)s'_{\overline{t}|}\frac{D_{x+t}}{D_x}\frac{(1+j)vd_{x+t} + (1+j)^2v^2d_{x+t+1} + \cdots + (1+j)^{n-t}v^{n-t}d_{x+n-1}}{l_{x+t}}\right\}$$

$$= \frac{D_{x+n}}{D_x} + \{\pi(1+\kappa)+c\}\left\{(1+j)\frac{M_x - M_{x+t}}{D_x} + (1+j)^2\frac{M_{x+1} - M_{x+t}}{D_x}\right.$$

$$\left. + \cdots + (1+j)^t \frac{M_{x+t-1} - M_{x+t}}{D_x} + (1+j)s'_{\overline{t}|}\frac{D_{x+t}}{D_x}\frac{M'_{x+t} - M'_{x+n}}{D'_{x+t}}\right\}$$

where $s'_{\overline{t}|}$ is calculated at rate j, while M'_{x+t}, M'_{x+n}, and D'_{x+t} are calculated at rate J, which is such that $\frac{1}{1+J} = \frac{1+j}{1+i}$

$$\text{Payment side} = \pi\frac{N_{x-1} - N_{x+t-1}}{D_x}$$

By equating the two sides π may be found, and $\pi' = \pi(1+\kappa)+c$.

23. Find the annual premium for a pure endowment payable at age $(x+r)$, the premiums alone to be returned if the life dies within s years, but the premiums to be returned with compound interest at rate j, if the life dies after s years.

$$\text{Benefit side} = \frac{D_{x+r}}{D_x} + \{\pi(1+\kappa)+c\}\frac{R_x - R_{x+s} - sM_{x+s}}{D_x}$$

$$+ \{\pi(1+\kappa)+c\}\left\{(1+j)s_{\overline{s+1}|}\frac{M_{x+s} - M_{x+r}}{D_x}\right.$$

$$+ (1+j)^{s+2}\frac{M_{x+s+1} - M_{x+r}}{D_x}$$

$$\left. + \cdots + (1+j)^r\frac{M_{x+r-1} - M_{x+r}}{D_x}\right\}$$

$$\text{Payment side} = \pi\frac{N_{x-1} - N_{x+r-1}}{D_x}$$

whence by equating, π may be found, and $\pi' = \pi(1+\kappa)+c$.

24. Find the net premium limited to 10 payments for an assurance on (x) under which the sum assured is 1 until all the premiums are paid up; thereafter the sum assured is the amount of the premiums accumulated at compound interest at rate i from the dates of payment of the premiums until the end of the year of death, should death happen before the expiration of the twentieth year of assurance. If the life survive 20 years, the sum assured again becomes 1, but the policy-holder is to be entitled to an annuity, until death, at rate i per annum upon the total amount of premiums actually paid without interest.

Benefit side

$$= \frac{M_x - M_{x+9}}{D_x} + \{\pi(1+\kappa)+c\}\{(1+i)+(1+i)^2 \cdots +(1+i)^{10}\}$$

$$\times \left\{\frac{C_{x+9} + (1+i)C_{x+10} + \cdots + (1+i)^{10}C_{x+19}}{D_x}\right\}$$

$$+ \frac{M_{x+20}}{D_x} + 10\{\pi(1+\kappa)+c\}i \, \frac{N_{x+20} + \frac{1}{2}M_{x+20}(1+i)^{\frac{1}{2}}}{D_x}$$

the expression "until death" being taken to imply a complete annuity.

$$\text{Payment side} = \pi \frac{N_{x-1} - N_{x+9}}{D_x}$$

And π may be found by equating the two sides.

25. Given values of a_x as follows: at 3 per cent., 19·895; at $3\frac{1}{2}$ per cent., 18·441; at 4 per cent., 17·155, find at 3 per cent. the annual premium for an assurance on (x) of £100, the sum assured to increase by £1 per annum for each year entered upon.

$$\text{Here P} = \frac{100A_x + (IA)_x}{1 + a_x}$$

To find $(IA)_x$ we may make use of the general formula established on page 297.

$$(IB)_x = -(1+i)\frac{dB_x}{di}$$

We have

Rate of Interest.	a_x	$A_x = 1-d(1+a_x)$	Δ	Δ^2
3	19·895	·39141		
			$-·04883$	
$3\frac{1}{2}$	18·441	·34258		·00798
			$-·04085$	
4	17·155	·30173		

$$\text{Then}\quad (IA)_x = -(1+i)\frac{dA_x}{di}$$

$$= -(1+i)\frac{\Delta A_x - \frac{1}{2}\Delta^2 A_x}{\Delta i}$$

$$= 1·03\,\frac{·04883 + ·00399}{·005}$$

$$= 10·881$$

$$P = \frac{39·141 + 10·881}{20·895}$$

$$= 2·394$$

26. If the office premium is 15 per cent. greater than the net premium, find at $3\frac{1}{2}$ per cent. the approximate annual office premium for an endowment assurance on a life aged 40 to be payable at age 60 or previous death, all the office premiums to be returned without interest in the event of death before age 60. Given $A^{1}_{40:\overline{20|}}$ at 3 per cent. $=·20413$, at $3\frac{1}{2}$ per cent. $=·19385$, at 4 per cent. $=·18428$; and $A_{40:\overline{20|}}$ at $3\frac{1}{2}$ per cent. $=·55338$, and $(1+{}_{|19}a_{40})$ at $3\frac{1}{2}$ per cent. $=13·207$.

The benefit side $= A_{40:\overline{20|}} + P(1·15)(IA)^{1}_{40:\overline{20|}}$.

By Lidstone's general formula for any benefit

$$(IB) = -(1+i)\frac{dB}{di}$$

$$\text{Hence}\quad (IA)^{1}_{40:\overline{20|}} = -(1+i)\frac{dA^{1}_{40:\overline{20|}}}{di}$$

$$= -(1·035)\frac{\Delta A^{1}_{40:\overline{20|}} - \frac{1}{2}\Delta^2 A^{1}_{40:\overline{20|}}}{·005}$$

$$= -(1·035)\frac{-·00957 - \frac{1}{2}\times ·00071}{·005}$$

$$= 2·054.$$

$$\begin{aligned}\text{Therefore the benefit} &= \cdot 55338 + \text{P}(1\cdot 15)2\cdot 054\\ &= \cdot 55338 + \text{P} \times 2\cdot 362.\end{aligned}$$

$$\begin{aligned}\text{The payment side} &= \text{P}(1 + a_{40:\overline{19}|})\\ &= \text{P} \times 13\cdot 207\end{aligned}$$

Equating the two sides we get

$$\begin{aligned}\text{P} \times 13\cdot 207 &= \cdot 55338 + \text{P} \times 2\cdot 362\\ \text{P} &= \frac{\cdot 55338}{10\cdot 845}\\ &= \cdot 05103\end{aligned}$$

The office premium therefore $= \cdot 05103 \times 1\cdot 15 = \cdot 05868$.

27. Use Lidstone's two approximate formulas to find by the $\text{O}^{[\text{M}]}$ table at $3\frac{1}{2}$ per cent. the annual premium for a joint-life endowment assurance on two lives aged 44 and 45 respectively, which shall increase by 1 per annum, *i.e.* 1 to be payable in the event of the first death taking place in the first year, 2 if it happen in the second, and so on, 20 if it happen in the twentieth, also 20 if both survive the twentieth year.

The two formulas are

$$\text{P}_{xy\overline{n}|} = \text{P}_{x\overline{n}|} + \text{P}_{y\overline{n}|} - \text{P}_{\overline{n}|}$$

and $$(\text{IA})_{xy\overline{n}|} = -(1+i)\frac{\Delta \text{A}_{xy\overline{n}|} - \frac{1}{2}\Delta^2 \text{A}_{xy\overline{n}|}}{\Delta i}$$

	3%	$3\frac{1}{2}$%	4%
$\text{P}_{44:\overline{20}|}$	·04391	·04206	·04030
$\text{P}_{45:\overline{20}|}$	·04438	·04254	·04078
	·08829	·08460	·08108
$\text{P}_{\overline{20}|}$	·03613	·03416	·03229
	·05216	·05044	·04879

Enter annual-premium conversion tables inversely to find

$a_{44:45:\overline{19}|}$	11·301	10·869	10·461

Enter single-premium conversion tables to find

$\text{A}_{44:45:\overline{20}|}$	·64172		·59864		55919
Δ		$-$·04308		$-$·03945	
Δ^2			·00363		

Hence approximately the annual premium

$$= \frac{-(1{\cdot}035)\dfrac{-{\cdot}03945-{\cdot}00182}{{\cdot}005}}{11{\cdot}869}$$
$$= {\cdot}720$$

28. A father, aged 35 next birthday, has a child, aged 1 next birthday. An assurance of £150 is to be paid on the child attaining the age of 21, provided the father be then alive. In the event of the father's death an annuity of £5 is to be paid annually until the child attains age 21, with a payment of £100 at that date. The premiums are to cease on the father's death, and to be returned on that of the child before reaching age 21. Deduce the formula for the annual premium.

$$\text{Benefit side} = 150\,\frac{D_{55:21}}{D_{35:1}} + 5 \times |_{20}a_{35}|_{1} + 100v^{20}{}_{20}p_{1}(1-{}_{20}p_{35})$$
$$+\{\pi(1+\kappa)+c\}\left(\frac{M_1-M_{21}}{D_1} + {}_1p_{35}\frac{M_2-M_{21}}{D_1} + \cdots + {}_{19}p_{35}\frac{M_{20}-M_{21}}{D_1}\right)$$
$$\text{Payment side} = \pi\,\frac{N_{34:0}-N_{54:20}}{D_{35:1}}$$

By equating the two sides, π may be found, and $\pi' = \pi(1+\kappa)+c$.

29. A debt is to be discharged by n equal annual payments to include (a) principal, (b) interest, and (c) the premium for an assurance to provide for the cessation of the annual payments and the extinction of the debt, if the debtor aged x die before the expiration of the n years. What should be the amount of the annual payment?

If the annual payments are made at the beginning of each of the n years, we have the annual equal payment of principal and interest to repay a loan of K in that time, $\frac{K}{a_{\overline{n}|}}$; and the annual premium to insure the balance of the loan outstanding is

$$\frac{\dfrac{K}{a_{\overline{n}|}}\{(M_x-M_{x+n-1})+v(M_x-M_{x+n-2})+\cdots+v^{n-2}(M_x-M_{x+1})\}}{N_{x-1}-N_{x+n-1}}$$

Adding this to $\frac{K}{a_{\overline{n}|}}$ we get the whole annual payment required.

Alternatively, we may proceed thus. If the loan be repayable by n equal instalments of $\frac{K}{a_{\overline{n}|}}$ the amount payable under the policy if (x) die in the first year is $\frac{K}{a_{\overline{n}|}}a_{\overline{n-1}|}$, if in the second $\frac{K}{a_{\overline{n}|}}a_{\overline{n-2}|}$, and so on. Therefore the whole benefit under the policy is

$$\frac{K}{a_{\overline{n}|}}\left(\frac{C_x a_{\overline{n-1}|} + C_{x+1}a_{\overline{n-2}|} + \cdot \cdot \cdot + C_{x+n-2}a_{\overline{1}|}}{D_x}\right)$$

and the premium to be added to the equal annual instalment of $\frac{K}{a_{\overline{n}|}}$ is

$$\frac{\frac{K}{a_{\overline{n}|}}(C_x a_{\overline{n-1}|} + C_{x+1}a_{\overline{n-2}|} + \cdot \cdot \cdot + C_{x+n-2}a_{\overline{1}|})}{N_{x-1} - N_{x+n-1}}$$

The two results are identical, for

$$\begin{aligned}
&C_x a_{\overline{n-1}|} + C_{x+1}a_{\overline{n-2}|} + \cdot \cdot \cdot + C_{x+n-2}a_{\overline{1}|} \\
= \; &C_x(1+v+v^2+ \cdot \cdot \cdot + v^{n-2}) + C_{x+1}(1+v+v^2+ \cdot \cdot \cdot + v^{n-3}) \\
&\qquad + \cdot \cdot \cdot + C_{x+n-2} \\
= \; &(C_x + C_{x+1} + \cdot \cdot \cdot + C_{x+n-2}) + v(C_x + C_{x+1} + \cdot \cdot \cdot + C_{x+n-3}) \\
&\qquad + \cdot \cdot \cdot + v^{n-2}C_x \\
= \; &(M_x - M_{x+n-1}) + v(M_x - M_{x+n-2}) + \cdot \cdot \cdot + v^{n-2}(M_x - M_{x+1})
\end{aligned}$$

30. Find the annual premium required at age x to secure £100 per annum for six years commencing on a child's sixteenth birthday; the premiums are to be limited to $(16-x)$, and in the event of death before age 16, the whole premiums paid are to be returned, while if it occur between ages 16 and 21 a proportion of the premiums paid, decreasing in arithmetical progression from five-sixths in the seventeenth year to one-sixth in the twenty-first, is to be returned.

Benefit side

$$= 100\frac{N_{15}-N_{21}}{D_x}+\{\pi(1+\kappa)+c\}\left\{\frac{R_x-R_{16}-(16-x)M_{16}}{D_x}\right.$$

$$\left.+(16-x)\{\pi(1+\kappa)+c\}\frac{\frac{5}{6}C_{16}+\frac{4}{6}C_{17}+\frac{3}{6}C_{18}+\frac{2}{6}C_{19}+\frac{1}{6}C_{20}}{D_x}\right\}$$

$$= 100\frac{N_{15}-N_{21}}{D_x}+\{\pi(1+\kappa)+c\}\left\{\frac{R_x-R_{16}-\frac{16-x}{6}(R_{16}-R_{22})}{D_x}\right\}$$

Payment side $= \pi\frac{N_{x-1}-N_{15}}{D_x}$

whence π may be found, and $\pi'=\pi(1+\kappa)+c$.

A better method of carrying through the transaction would be to have six policies each for £100 payable at ages 16, 17, 18, 19, 20, and 21 respectively, with return of premiums paid in the event of previous death.

31. Find the single payment required to redeem future premiums under a child's endowment with returnable premiums, effected at age x, and payable at age $(x+t)$, which has been n years in force.

If π and π' are the net and office premiums for the original benefit, and if we write A and A′ for the net and office single payments now required, we may take it that A must be equal to the value of the future premiums now to be forgone less the return in respect of these premiums, and plus the return in respect of the single payment. That is

$$A = \pi\frac{N_{x+n-1}-N_{x+t-1}}{D_{x+n}}-\pi'\frac{R_{x+n}-R_{x+t}-(t-n)M_{x+t}}{D_{x+n}}$$

$$+\{A(1+\kappa')+c'\}\frac{M_{x+n}-M_{x+t}}{D_{x+n}}$$

Hence

$$A = \frac{\pi(N_{x+n-1}-N_{x+t-1})-\pi'\{R_{x+n}-R_{x+t}-(t-n)M_{x+t}\}+c'(M_{x+n}-M_{x+t})}{D_{x+n}-(1+\kappa')(M_{x+n}-M_{x+t})}$$

and the single premium required

$$A' = A(1+\kappa')+c'.$$

32. If you were furnished with the office annual premiums for temporary assurances of £100 for from one to fifteen years at age x, and also with full tables of annuities, how would you arrive at the annual premium for an assurance of £1000 decreasing by one-fifteenth each year, and at a premium also decreasing by one-fifteenth annually?

Let the premium required commence at 15P and decrease by P per annum.

Benefit side

$$= 1000 \times \tfrac{1}{15}(P'^{1}_{x:\overline{1}|} + P'^{1}_{x:\overline{2}|}a_{x:\overline{2}|} + P'^{1}_{x:\overline{3}|}a_{x:\overline{3}|} + \cdots + P'^{1}_{x:\overline{15}|}a_{x:\overline{15}|})$$

where $P'^{1}_{x:\overline{1}|}$, etc., are the office annual premiums per unit.

Payment side $= P(a_{x:\overline{1}|} + a_{x:\overline{2}|} + a_{x:\overline{3}|} + \cdots + a_{x:\overline{15}|})$

and P may be obtained therefrom.

A more practical method of carrying through the assurance, which, however, would not fulfil the condition that the premium should decrease by an equal amount each year, would be to get (x) to effect fifteen policies, as follows:—

Term.	Sum Assured.	Premium payable at beginning of each year during term.
1	$\frac{1000}{15}$	$\frac{1000}{15}P'^{1}_{x:\overline{1}|}$
2	$\frac{1000}{15}$	$\frac{1000}{15}P'^{1}_{x:\overline{2}|}$
3	$\frac{1000}{15}$	$\frac{1000}{15}P'^{1}_{x:\overline{3}|}$
⋮	⋮	⋮
15	$\frac{1000}{15}$	$\frac{1000}{15}P'^{1}_{x:\overline{15}|}$

The sum assured decreases in the required manner, but the premium does not, though it too decreases in total, since at the end of each year there is always one premium less to pay out of the fifteen with which we started.

CHAPTER XVII

Successive Lives

EXAMPLES

1. Give an explanation of the expressions

$$(a)\ A_x A_y A_z$$

$$(b)\ (1+i)^{\frac{3}{2}} A_x A_y A_z$$

(*a*) This represents the present value of 1 to be paid at the end of the year of death of (z), who is to be nominated at the end of the year of death of (y), who is to be nominated at the end of the year of death of (x).

(*b*) This represents the present value of 1 to be paid at the moment of death of (z), who is to be nominated at the moment of death of (y), who is to be nominated at the moment of death of (x).

2. Show that the single premium for an assurance, with return of the premium along with the sum assured, is equal to the value of all the future fines on successive lives, where the lives are to be nominated all of the same age as that at present of the life in possession. Explain verbally why this should be so.

The single premium for an assurance with return of premium is found as follows:—

$$\text{Benefit side} = A_x + B \times A_x$$

$$\text{Payment side} = B$$

$$\text{and}\quad B = \frac{A_x}{1 - A_x}$$

Again, the value of all future fines on successive lives of equal ages is

$$A_x + (A_x)^2 + (A_x)^3 + \cdot\ \cdot\ \cdot\ ad\ inf.$$

$$= \frac{A_x}{1 - A_x}$$

Now, under the assurance, we obtain $1+B$ at the death of (x), whereof 1 will pay the fine falling due at his death, and B will set up a new policy of like nature on a life then aged x, which in turn will pay the fine at the second death, and provide B for a third policy, and so on, *ad infinitum.*

3. Find the value of an annuity during four successive lives—the second, third, and fourth lives being nominated on the deaths of the first, second, and third lives respectively.

$$\begin{aligned} a_{\overline{w(x)(y)(z)}} &= a_w + A_w(1+a_x) + A_wA_x(1+a_y) + A_wA_xA_y(1+a_z) \\ &= \frac{1-A_w}{d} - 1 + A_w\frac{1-A_x}{d} + A_wA_x\frac{1-A_y}{d} + A_wA_xA_y\frac{1-A_z}{d} \\ &= \frac{1-A_wA_xA_yA_z}{d} - 1 \end{aligned}$$

4. A copyhold estate is held on two lives, each renewable at the end of the year in which it drops, by a life aged 10, on payment of a fine of £8. Assuming the two lives to be now aged 30 and 35 respectively, find the present value of all the fines in perpetuity at 3 per cent. interest.

The value of the fines payable on the succession of lives starting from the life now aged 30 is

$$\begin{aligned} &8(A_{30} + A_{30}\times A_{10} + A_{30}\times A_{10}\times A_{10} + \cdots) \\ &\quad = 8A_{30}\{1 + A_{10} + (A_{10})^2 + \cdots\} \\ &\quad = \frac{8A_{30}}{1-A_{10}} \end{aligned}$$

Similarly, on the succession starting from the life aged 35 we have

$$\frac{8A_{35}}{1-A_{10}}$$

The present value of all the fines is therefore

$$\frac{8(A_{30}+A_{35})}{1-A_{10}}$$

Taking the Carlisle table at 3 per cent. we have

$$\begin{aligned} \frac{8(\cdot40129+\cdot43399)}{1-\cdot28606} &= \frac{6\cdot68224}{\cdot71394} \\ &= 9\cdot360 \end{aligned}$$

5. C and D have the perpetual right of alternate presentation to a living of the annual value of £k. Assuming that D has the next right of presentation, that the present incumbent is aged y, and that his successors will all be aged x on appointment, find the sum which C should pay D to purchase his interest.

On the principles of *Text Book,* Article 32, the value of the perpetual right is

$$k\{\dot{a}_y + \mathrm{A}_y(\tfrac{1}{2}+\dot{a}_x) + \mathrm{A}_y\mathrm{A}_x(\tfrac{1}{2}+\dot{a}_x) + \mathrm{A}_y(\mathrm{A}_x)^2(\tfrac{1}{2}+\dot{a}_x) + \cdot\ \cdot\ \cdot\ \}$$

Of this series the even terms represent D's share, which is therefore equal to

$$k\mathrm{A}_y(\tfrac{1}{2}+\dot{a}_x)\{1 + (\mathrm{A}_x)^2 + (\mathrm{A}_x)^4 + \cdot\ \cdot\ \cdot\ \}$$

$$= \frac{k\mathrm{A}_y(\tfrac{1}{2}+\dot{a}_x)}{1-(\mathrm{A}_x)^2}$$

Modifying this formula in accordance with *Text Book,* Article 35, *i.e.* assuming a new presentation to be made at the moment of death, and the presentee to then enter upon a continuous annuity, we have

$$\frac{k\bar{\mathrm{A}}_y\bar{a}_x}{1-(\bar{\mathrm{A}}_x)^2}$$

CHAPTER XVIII

Policy-Values

1. The PROSPECTIVE and RETROSPECTIVE VALUES of policies under the various ordinary classes of assurance should be written down and their identity proved. We propose to deduce the retrospective values for whole-life and endowment assurance policies, and prove that they are equal to the corresponding prospective values.

Assume that each of l_x persons has effected a whole-life policy at an annual premium of P_x. Then the premiums paid at the beginning of each year accumulated at rate i to the end of n years amount to

$$P_x\{l_x(1+i)^n + l_{x+1}(1+i)^{n-1} + \cdot \cdot \cdot + l_{x+n-1}(1+i)\}$$

$$= P_x \frac{D_x + D_{x+1} + \cdot \cdot \cdot + D_{x+n-1}}{v^{x+n}}$$

$$= P_x \frac{N_{x-1} - N_{x+n-1}}{v^{x+n}}$$

The claims paid at the end of each year accumulated at rate i to the end of n years amount to

$$d_x(1+i)^{n-1} + d_{x+1}(1+i)^{n-2} + \cdot \cdot \cdot + d_{x+n-1}$$

$$= \frac{C_x + C_{x+1} + \cdot \cdot \cdot + C_{x+n-1}}{v^{x+n}}$$

$$= \frac{M_x - M_{x+n}}{v^{x+n}}$$

The fund in hand made up of the surplus of accumulated premiums over accumulated claims is therefore

$$P_x \frac{N_{x-1} - N_{x+n-1}}{v^{x+n}} - \frac{M_x - M_{x+n}}{v^{x+n}}$$

Dividing this by the number of survivors, viz. l_{x+n}, we have the average sum in hand in respect of each survivor, that is

$$_nV_x = \frac{P_x(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}}$$

which is the policy-value found retrospectively.

The prospective value is

$$_nV_x = A_{x+n} - P_x(1 + a_{x+n})$$

To prove the values by the two methods equal, we must assume the rates of mortality and interest to be the same as those employed in the calculation of P_x.

$$\text{Now} \quad \frac{P_x(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}}$$

$$= \frac{P_xN_{x-1} - M_x}{D_{x+n}} + \frac{M_{x+n} - P_xN_{x+n-1}}{D_{x+n}}$$

$$= A_{x+n} - P_x(1 + a_{x+n})$$

$$\text{since} \quad P_x = \frac{M_x}{N_{x-1}}$$

For the endowment assurance payable at age $(x+t)$ or previous death, the value by the prospective method is

$$_nV_{\overline{xt|}} = A_{x+n:\overline{t-n|}} - P_{\overline{xt|}}(1 + a_{x+n:\overline{t-n-1|}})$$

By the retrospective method the value will be the same as for the whole-life policy with the substitution of $P_{\overline{xt|}}$ for P_x. Thus

$$_nV_{\overline{xt|}} = \frac{P_{\overline{xt|}}(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}}$$

$$\text{Now} \quad \frac{P_{\overline{xt|}}(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}}$$

$$= \frac{P_{\overline{xt|}}(N_{x-1} - N_{x+t-1}) - (M_x - M_{x+t} + D_{x+t})}{D_{x+n}}$$

$$+ \frac{(M_{x+n} - M_{x+t} + D_{x+t}) - P_{\overline{xt|}}(N_{x+n-1} - N_{x+t-1})}{D_{x+n}}$$

$$= A_{x+n:\overline{t-n|}} - P_{\overline{xt|}}(1 + a_{x+n:\overline{t-n-1|}})$$

$$\text{since} \quad P_{\overline{xt|}} = \frac{M_x - M_{x+t} + D_{x+t}}{N_{x-1} - N_{x+t-1}}$$

Others which should be worked out similarly are:—

Pure endowment

$$ {}_nV_{\overset{1}{x\overline{t}|}} = A_{x+n:\overset{\;\;1}{\overline{t-n}|}} - P_{\overset{\;\;1}{x\overline{t}|}}(1+a_{x+n:\overline{t-n-1}|}) \quad \text{prospectively} $$

$$ = \frac{P_{\overset{\;\;1}{x\overline{t}|}}(N_{x-1}-N_{x+n-1})}{D_{x+n}} \quad \text{retrospectively} $$

Temporary assurance

$$ {}_nV^1_{x\overline{t}|} = A^{\;1}_{x+n:\overline{t-n}|} - P^1_{x\overline{t}|}(1+a_{x+n:\overline{t-n-1}|}) \quad \text{prospectively} $$

$$ = \frac{P^1_{x\overline{t}|}(N_{x-1}-N_{x+n-1})-(M_x-M_{x+n})}{D_{x+n}} \quad \text{retrospectively} $$

Limited-payment policy

(1) When $n < t$

$$ {}_{n:t}V_x = A_{x+n} - {}_tP_x(1+a_{x+n:\overline{t-n-1}|}) \quad \text{prospectively} $$

$$ = \frac{{}_tP_x(N_{x-1}-N_{x+n-1})-(M_x-M_{x+n})}{D_{x+n}} \quad \text{retrospectively} $$

(2) When n = or $> t$

$$ {}_{n:t}V_x = A_{x+n} \quad \text{prospectively} $$

$$ = \frac{{}_tP_x(N_{x-1}-N_{x+t-1})-(M_x-M_{x+n})}{D_{x+n}} \quad \text{retrospectively} $$

Joint-life assurance

$$ {}_nV_{xy} = A_{x+n:y+n} - P_{xy}(1+a_{x+n:y+n}) \quad \text{prospectively} $$

$$ = \frac{P_{xy}(N_{x-1:y-1}-N_{x+n-1:y+n-1})-(M_{xy}-M_{x+n:y+n})}{D_{x+n:y+n}} \quad \text{retrospectively} $$

Leasehold assurance

$$ {}_nV_{\overline{t}|} = v^{t-n} - P_{\overline{t}|}(1+a_{\overline{t-n-1}|}) \quad \text{prospectively} $$

$$ = P_{\overline{t}|}(1+i)s_{\overline{n}|} \quad \text{retrospectively} $$

Further problems similar to these will occur later, and will then be disposed of.

2. The notation for policy-values in regard to select tables follows the rules already laid down (see page 140).

If $t < n$, the period during which selection is assumed to have effect,

$$\begin{aligned}
{}_tV_{[x]} &= A_{[x]+t} - P_{[x]}a_{[x]+t} \\
&= 1 - \frac{a_{[x]+t}}{a_{[x]}}
\end{aligned}$$

If $t > n$

$$\begin{aligned}
{}_tV_{[x]} &= A_{x+t} - P_{[x]}a_{x+t} \\
&= 1 - \frac{a_{x+t}}{a_{[x]}}
\end{aligned}$$

If the life presently aged $(x+t)$ is assumed to be still select the reserve value is

$$A_{[x+t]} - P_{[x]}a_{[x+t]} = 1 - \frac{a_{[x+t]}}{a_{[x]}}$$

3. A very simple proof of *Text Book* formula (3) is as follows:—

$$\begin{aligned}
{}_nV_x + P_x &= A_{x+n} - P_x a_{x+n} \\
&= (vq_{x+n} + vp_{x+n}A_{x+n+1}) - P_x vp_{x+n}(1 + a_{x+n+1}) \\
&= vq_{x+n} + vp_{x+n}\{A_{x+n+1} - P_x(1 + a_{x+n+1})\} \\
&= v(q_{x+n} + p_{x+n} \times {}_{n+1}V_x)
\end{aligned}$$

Or, retrospectively,

$$\begin{aligned}
{}_nV_x + P_x &= \frac{P_x(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}} + P_x \\
&= \frac{P_x(N_{x-1} - N_{x+n}) - (M_x - M_{x+n+1})}{D_{x+n}} + \frac{C_{x+n}}{D_{x+n}} \\
&= vp_{x+n}\frac{P_x(N_{x-1} - N_{x+n}) - (M_x - M_{x+n+1})}{D_{x+n+1}} + vq_{x+n} \\
&= v(q_{x+n} + p_{x+n} \times {}_{n+1}V_x)
\end{aligned}$$

Similarly, we have for pure endowments

$$
\begin{aligned}
{}_nV_{x\overset{1}{\overline{t|}}} + P_{x\overset{1}{\overline{t|}}} &= A_{x+n:\overset{1}{\overline{t-n|}}} - P_{x\overset{1}{\overline{t|}}}\,a_{x+n:\overline{t-n-1|}} \\
&= vp_{x+n}A_{x+n+1:\overset{1}{\overline{t-n-1|}}} - P_{x\overset{1}{\overline{t|}}}\,vp_{x+n}(1+a_{x+n+1:\overline{t-n-2|}}) \\
&= vp_{x+n}\{A_{x+n+1:\overset{1}{\overline{t-n-1|}}} - P_{x\overset{1}{\overline{t|}}}(1+a_{x+n+1:\overline{t-n-2|}})\} \\
&= vp_{x+n}\times{}_{n+1}V_{x\overset{1}{\overline{t|}}}
\end{aligned}
$$

Or, again, $$
\begin{aligned}
{}_nV_{x\overset{1}{\overline{t|}}} + P_{x\overset{1}{\overline{t|}}} &= \frac{P_{x\overset{1}{\overline{t|}}}(N_{x-1}-N_{x+n-1})}{D_{x+n}} + P_{x\overset{1}{\overline{t|}}} \\
&= vp_{x+n}\times\frac{P_{x\overset{1}{\overline{t|}}}(N_{x-1}-N_{x+n})}{D_{x+n+1}} \\
&= vp_{x+n}\times{}_{n+1}V_{x\overset{1}{\overline{t|}}}
\end{aligned}
$$

4. By *Text Book* formula (3),

$$
\begin{aligned}
{}_nV_x + P_x &= v(q_{x+n} + p_{x+n}\times{}_{n+1}V_x) \\
&= v\{q_{x+n} + (1-q_{x+n})\,{}_{n+1}V_x\} \\
&= v\{q_{x+n}(1-{}_{n+1}V_x) + {}_{n+1}V_x\}
\end{aligned}
$$

Therefore $({}_nV_x + P_x)(1+i) = q_{x+n}(1-{}_{n+1}V_x) + {}_{n+1}V_x$

From this we see that the reserve value at the beginning of the year, and the premium then paid, both accumulated to the end of the year, are equal to the reserve value at the end of the year, together with a contribution towards the claims payable.

Now, if the mortality actually experienced agree with that assumed, the account will work out as follows for l_{x+n} policies, each for 1, existing at the commencement of the year:—

Claims payable . . . . $= d_{x+n}$

Contribution towards claims payable

$= l_{x+n}\times q_{x+n}(1-{}_{n+1}V_x)$. $= d_{x+n}(1-{}_{n+1}V_x)$

Reserve released . . $= d_{x+n}\times{}_{n+1}V_x$

Together . . . . d_{x+n}

so that the claims payable are exactly met.

Profit or loss from mortality in an Insurance Office therefore depends on a comparison of the claims payable less the reserve

released, with the expected contribution towards claims payable. If the claims, less the reserve, exceed the contribution, there is a loss to the office, and *vice versa*.

Therefore mere comparison of the claims payable with the accumulations of premiums received is no test of profit or loss from mortality. It is impossible to view the contracts in this light, since at the date of the claim the accumulations of premiums received are not available, having been applied to pay current claims and to increase the reserve (apart from payment of expenses and distribution of profit). Further, however long the life live, even beyond his expectation at date of entry, there is a loss to the office in respect of his claim (unless indeed he live beyond the limiting age of the mortality table used), since ${}_nV_x$ is the average reserve in hand at any time while the claim to be paid is 1. Every life assured on the books has to contribute towards the deficit, and the profit or loss on mortality, as already stated, depends on how this deficit compares with the contribution.

In the case of an assurance under which no further premiums are payable, we have

$$A_{x+n}(1+i) = q_{x+n}(1-A_{x+n+1})+A_{x+n+1}$$

Here the reserve value accumulated for a year is equal to the reserve value at the end of the year, together with a contribution towards the claims payable.

In the case of an annuity, we have

$$a_{x+n}(1+i) = (1+a_{x+n+1})-q_{x+n}(1+a_{x+n+1})$$

This formula was discussed in the notes on Chapter VII., page 120.

5. Returning to the equation

$$({}_nV_x+P_x)(1+i) = q_{x+n}(1-{}_{n+1}V_x)+{}_{n+1}V_x$$

we see that

$${}_nV_x+P_x > = < {}_{n+1}V_x$$

according as

$$({}_nV_x+P_x)i < = > q_{x+n}(1-{}_{n+1}V_x)$$

Now, $({}_nV_x+P_x)$ is the reserve value immediately after payment of the $(n+1)$th premium and ${}_{n+1}V_x$ the reserve value immediately before payment of the $(n+2)$th premium. Therefore the former is $> = <$ the latter, according as interest on the reserve is $< = >$ the current mortality risk.

It will be found that, unless the policy has been a considerable number of years in force, $({}_nV_x + P_x)i < q_{x+n}(1 - {}_{n+1}V_x)$, and therefore ${}_nV_x + P_x > {}_{n+1}V_x$. On the other hand, where the policy has been a long period in force $({}_nV_x + P_x)i$ may exceed $q_{x+n}(1 - {}_{n+1}V_x)$, and therefore ${}_nV_x + P_x < {}_{n+1}V_x$, that is the reserve value of the policy will increase in the course of the year.

In the case of a temporary assurance

$$ {}_nV^1_{x\overline{t}|} + P^1_{x\overline{t}|} > = < {}_{n+1}V^1_{x\overline{t}|} $$

according as

$$ ({}_nV^1_{x\overline{t}|} + P^1_{x\overline{t}|})i < = > q_{x+n}(1 - {}_{n+1}V^1_{x\overline{t}|}) $$

And it will be found (where t is not very great) that interest on the reserve is insufficient to provide for the current mortality risk, and that ${}_nV^1_{x\overline{t}|} + P^1_{x\overline{t}|}$ is greater than ${}_{n+1}V^1_{x\overline{t}|}$. Indeed ${}_nV^1_{x\overline{t}|}$ alone frequently exceeds ${}_{n+1}V^1_{x\overline{t}|}$ as may be seen from *Text Book*, page 319, Table C.

In the case of a pure endowment

$$ {}_nV_{x\overline{t}|}^{\;\;1} + P_{x\overline{t}|}^{\;\;1} > = < {}_{n+1}V_{x\overline{t}|}^{\;\;1} $$

according as

$$ ({}_nV_{x\overline{t}|}^{\;\;1} + P_{x\overline{t}|}^{\;\;1})i < = > -q_{x+n} \times {}_{n+1}V_{x\overline{t}|}^{\;\;1} $$

But obviously

$$ ({}_nV_{x\overline{t}|}^{\;\;1} + P_{x\overline{t}|}^{\;\;1})i > -q_{x+n} \times {}_{n+1}V_{x\overline{t}|}^{\;\;1} $$

since the latter is negative.

Therefore ${}_nV_{x\overline{t}|}^{\;\;1} + P_{x\overline{t}|}^{\;\;1} < {}_{n+1}V_{x\overline{t}|}^{\;\;1}$, showing that the reserve value in this class increases in the course of the year.

For the endowment assurance (the summation of the last two) we have

$$ {}_nV_{x\overline{t}|} + P_{x\overline{t}|} > = < {}_{n+1}V_{x\overline{t}|} $$

according as

$$ ({}_nV_{x\overline{t}|} + P_{x\overline{t}|})i < = > q_{x+n}(1 - {}_{n+1}V_{x\overline{t}|}) $$

that is, according as the interest on the reserve is $< = >$ the current mortality risk.

6. As an illustration of how to ascertain profit or loss from mortality, let us assume in connection with the example worked out on page 316 of the *Text Book* that the actual mortality experienced for the first five years was as shown below, and not according to the *Text Book* table, and that the annual premium was ·01873.

Age.	l_x	d_x
30	89685	655
31	89030	740
32	88290	700
33	87590	720
34	86870	750
35	86120	765
etc.	etc.	etc.

We have first to construct a table, thus :—

Year.	Premiums received.	${}_nV_{30}$ as per *Text Book*, p. 316.	Survivors at end of year.	Fund required at end of year.
1	1679·800	·01168	89030	1039·870
2	1667·532	·02364	88290	2087·176
3	1653·672	·03590	87590	3144·481
4	1640·561	·04847	86870	4210·589
5	1627·075	·06133	86120	5281·740
etc.	etc.	etc.	etc.	etc.

Then we may find the profit or loss, as follows :—

Year.	1.	2.	3.	4.	5.
Fund at beginning of year .	0	1039·870	2087·176	3144·481	4210·589
Premiums received . .	1679·800	1667·532	1653·672	1640·561	1627·075
	1679·800	2707·402	3740·848	4785·042	5837·664
Interest	50·394	81·222	112·225	143·551	175·130
	1730·194	2788·624	3853·073	4928·593	6012·794
Claims	655	740	700	720	750
Fund in hand at end of year	1075·194	2048·624	3153·073	4208·593	5262·794
Fund required do.	1039·870	2087·176	3144·481	4210·589	5281·740
Profit (+) or Loss (−). .	+35·324	−38·552	+8·592	−1·996	−18·946

7. By *Text Book* formula (10)

$$ {}_nV_x = \frac{A_{x+n} - A_x}{1 - A_x} $$

This result may be proved by general reasoning.

Suppose (x) to enter into such a contract as that described in Article 60 of *Text Book*, Chapter VII., under which the amount payable after deduction of the single premium shall be 1. That is, if B be the single premium the total sum assured is $1 + B$, the loan on the policy is B, and the interest payable in advance on this loan is dB. Then $B = A_x(1 + B)$

whence $B = \dfrac{A_x}{1 - A_x}$

The sum assured is therefore . . . $\dfrac{1}{1 - A_x}$

The single premium, which is also the amount of the loan, is . . . . . . $\dfrac{A_x}{1 - A_x}$

The yearly interest payable in advance, which is also the annual premium for the net amount payable, is . . . . . . $\dfrac{dA_x}{1 - A_x}$

The net amount payable at death is the sum assured, less the loan, that is $\dfrac{1}{1 - A_x} - \dfrac{A_x}{1 - A_x}$, or . 1.

Now, after n years the reserve required under the above single-premium policy for $\dfrac{1}{1 - A_x}$ is . $\dfrac{A_{x+n}}{1 - A_x}$

from which deduct the policy loan which the office must take credit for . . . . . $\dfrac{A_x}{1 - A_x}$

The difference is . . . . . . $\dfrac{A_{x+n} - A_x}{1 - A_x}$

which is the reserve required for an annual-premium policy under which the net amount payable is 1, and which has been n years in force. That is,

$$ {}_nV_x = \frac{A_{x+n} - A_x}{1 - A_x} $$

8. The proof of *Text Book,* Articles 41 to 43, seems rather laboured. The matter may be put more shortly.

$$
\begin{aligned}
{}_nV'_x &= \frac{P'_{x+n} - P'_x}{P'_{x+n} + d} \\
&= \frac{(P_{x+n} - P_x)(1+\kappa)}{P_{x+n}(1+\kappa) + c + d} \\
&= \frac{(P_{x+n} - P_x)(1+\kappa)}{(P_{x+n} + d)(1+\kappa) + (c - \kappa d)} \\
&= \frac{{}_nV_x}{1 + \dfrac{c - \kappa d}{(P_{x+n} + d)(1+\kappa)}}
\end{aligned}
$$

Now, according as $c > = < \kappa d$, the expression $\dfrac{c - \kappa d}{(P_{x+n} + d)(1+\kappa)}$ is positive, zero, or negative; and ${}_nV'_x < = > {}_nV_x$.

Or, again, $${}_nV'_x = \frac{P_{x+n} - P_x}{P_{x+n} + \dfrac{c+d}{1+\kappa}}$$

Therefore $${}_nV'_x > = < {}_nV_x$$

as $$\frac{P_{x+n} - P_x}{P_{x+n} + \dfrac{c+d}{1+\kappa}} > = < \frac{P_{x+n} - P_x}{P_{x+n} + d}$$

that is, as $$d > = < \frac{c+d}{1+\kappa}$$

or as $$d - \frac{d}{1+\kappa} > = < \frac{c}{1+\kappa}$$

or, finally, as $$\kappa d > = < c$$

9. *Text Book,* Article 48, is very important, and has a wide bearing in a consideration of the effect that an increase in the mortality has on policy-values.

One is apt to assume that merely because one mortality table exhibits higher rates of mortality than another, the former requires

larger reserves than the latter. But it is impossible to argue so fast; for we see that, if in the expression

$$1-\frac{P_x+d}{P_{x+n}+d}$$

we increase both P_x and P_{x+n}, we cannot tell whether the whole expression is increased or diminished without more minute examination.

The same point is seen on examination of the value of ${}_nV_x$ found retrospectively,

$$\frac{1}{l_{x+n}}\Big[P_x\Big\{l_x(1+i)^n+l_{x+1}(1+i)^{n-1}+\cdots+l_{x+n-1}(1+i)\Big\}$$
$$-\Big\{d_x(1+i)^{n-1}+d_{x+1}(1+i)^{n-2}+\cdots+d_{x+n-1}\Big\}\Big]$$

If the rate of mortality as a whole is increased, it is quite true that P_x is increased and l_{x+n} decreased, both tending to increase ${}_nV_x$, but at the same time l_{x+1}, l_{x+2}, etc., are decreased, and d_x, d_{x+1}, etc., proportionally increased, all tending to decrease ${}_nV_x$, and the final result may quite well be a lower value for ${}_nV_x$.

Now, as pointed out in *Text Book*, Article 48, "If the increase be proportionally greater at the younger ages, the policy-value will be diminished, and if the increase be proportionally greater at the older ages, the policy-value will be augmented." Also, Dr T. B. Sprague, in his paper, "How does an increased mortality affect policy-values" (*J. I. A.*, xxi. 109), says: "It seems that we may fairly draw the following conclusions:—

(1) If two tables show the same mortality at young ages and at higher ages an increasing difference in the rate of mortality, then the one which shows the higher rate of mortality will require larger policy-values.

(2) If two tables show the same mortality at high ages, but an increasing divergence as we proceed to younger ages, then the table which shows the lower mortality at younger ages will require larger policy-values.

(3) If two tables, A and B, show the same rate of mortality at the middle ages, say about 50, but at younger ages the table A shows the higher mortality and at higher ages the lower mortality, then table A will require the lower policy-values."

Finally, Dr Sprague tells us that "policy-values do not at all depend upon the absolute rate of mortality exhibited, but only

upon the progression that the rate of mortality exhibits. It is the table in which the mortality increases the more rapidly, that requires the larger policy-values."

10. As shown in *Text Book*, Article 49.

$$_nV'_x > = < {}_nV_x$$

according as
$$\frac{1+a'_x}{1+a_x} > = < \frac{1+a'_{x+n}}{1+a_{x+n}}$$

If, then, it be desired to compare the reserves at 3 per cent. of the H^M experience with those of the O^M, a table should be worked out for each age as follows:—

Ratio of annuities-due, $\frac{H^M}{O^M}$, at 3 per cent. for comparison of policy-values.

Age.	Ratio $\frac{H^M a_x}{O^M a_x}$
20	·9720
25	·9778
30	·9813
35	·9850
40	·9881
45	·9888
50	·9895
55	·9890
60	·9870

Here then $$H^M{}_{20}V_{20} < O^M{}_{20}V_{20}$$

because $$\frac{H^M a_{20}}{O^M a_{20}} < \frac{H^M a_{40}}{O^M a_{40}}$$

Also $$H^M{}_{15}V_{40} < O^M{}_{15}V_{40}$$

because $$\frac{H^M a_{40}}{O^M a_{40}} < \frac{H^M a_{55}}{O^M a_{55}}$$

But $$H^M{}_{20}V_{40} > O^M{}_{20}V_{40}$$

because $$\frac{H^M a_{40}}{O^M a_{40}} > \frac{H^M a_{60}}{O^M a_{60}}$$

The matter becomes rather more complicated when it is desired to compare the H^M and $H^{M(5)}$ reserves with the O^M.

In this case when $n < 5$

$$H^M{}_nV_x > = < O^M{}_nV_x$$

according as
$$\frac{H^M a_x}{O^M a_x} > = < \frac{H^M a_{x+n}}{O^M a_{x+n}}$$

But when n = or > 5,

$$H^M \text{ and } H^{M(5)}{}_nV_x > = < O^M{}_nV_x$$

according as
$$1 - \frac{H^{M(5)} a_{x+n}}{H^M a_x} > = < 1 - \frac{O^M a_{x+n}}{O^M a_x}$$

that is, as
$$\frac{H^M a_x}{O^M a_x} > = < \frac{H^{M(5)} a_{x+n}}{O^M a_{x+n}}$$

Having drawn up a table of the two ratios $\dfrac{H^M a_x}{O^M a_x}$ and $\dfrac{H^{M(5)} a_{x+n}}{O^M a_{x+n}}$ we can tell, by inspection of the ratios, at what ages at entry and for what terms the H^M and $H^{M(5)}$ reserves will be greater than, equal to, or less than the O^M reserves.

It is wrong to assume that, because one table or combination of tables shows larger reserves than another for whole-life policies, the same relation will hold for endowment assurances. Comparison must be instituted between an entirely different set of functions, viz., term annuities. For

$${}_nV'_{x\overline{t}|} > = < {}_nV_{x\overline{t}|}$$

according as
$$\frac{a'_{x\overline{t}|}}{a_{x\overline{t}|}} > = < \frac{a'_{x+n:\overline{t-n}|}}{a_{x+n:\overline{t-n}|}}$$

This is not a merely theoretical point; for as a matter of fact, while the H^M and $H^{M(5)}$ reserves are greater on the whole than the O^M for whole-life assurances, the reverse is the case for endowment assurances.

If two mortality tables yield equal policy-values, that is, if

${}_n\mathrm{V}'_x = {}_n\mathrm{V}_x$, then the ratio $\dfrac{\mathrm{a}'_x}{\mathrm{a}_x}$ will be constant for all values of x, and we may write

$$\frac{1+a'_x}{1+a_x} = \frac{1}{1+\kappa}$$

whence
$$a'_x = \frac{1+a_x}{1+\kappa} - 1$$

Then
$$\begin{aligned}
{}_n\mathrm{V}'_x &= 1 - \frac{1+a'_{x+n}}{1+a'_x} \\
&= 1 - \frac{\dfrac{1+a_{x+n}}{1+\kappa}}{\dfrac{1+a_x}{1+\kappa}} \\
&= 1 - \frac{1+a_{x+n}}{1+a_x} \\
&= {}_n\mathrm{V}_x \text{ as required.}
\end{aligned}$$

But again
$$\begin{aligned}
p'_x &= \frac{a'_x}{v(1+a'_{x+1})} \\
&= \frac{\dfrac{1+a_x}{1+\kappa} - 1}{v\,\dfrac{1+a_{x+1}}{1+\kappa}} \\
&= \frac{a_x - \kappa}{v(1+a_{x+1})} \\
&= \frac{a_x - \kappa}{\dfrac{a_x}{p_x}} \\
&= p_x\left(1 - \frac{\kappa}{a_x}\right)
\end{aligned}$$

which is *Text Book* formula (29).

The conclusions of *Text Book*, Articles 59 and 60, may be proved directly for the rate of mortality, q'_x.

If
$$p'_x = p_x\left(1 - \frac{\kappa}{a_x}\right)$$
$$q'_x = 1 - p_x + \frac{\kappa p_x}{a_x}$$
$$= q_x + \frac{\kappa}{v(1+a_{x+1})}$$

The addition to be made to the rate of mortality at age x if equal policy-values are to be produced is therefore $\dfrac{\kappa}{v(1+a_{x+1})}$ which increases with an increase in x.

Now, if instead of this increasing function we make a constant addition of r to the rate of mortality, the increase will not be so rapid as is required to give equal policy-values. Therefore, on the principles of *Text Book*, Article 48, the effect of adding a constant to q_x is to diminish policy-values.

Again, if for $\dfrac{\kappa}{v(1+a_{x+1})}$ we substitute $-r$, the rate of mortality will be increasing more rapidly in proportion than under the formula which produces equal policy-values; and therefore on the principles of *Text Book*, Article 48, the effect of deducting a constant from q_x is to increase policy-values.

11. In this connection it will be useful to discuss the reserves required for policies upon lives subject to extra mortality. Extra mortality will probably occur in one or other of three well-defined ways.

(1) The extra mortality may be higher than the normal throughout, the difference being small and slowly increasing at first, but becoming great in the later years of insurance. We should expect reserves under such a table to be greater than those under the normal, as the increase in the mortality is proportionately greater at the older ages.

(2) If the extra mortality is greater than the normal, but by a constant difference throughout, we should expect the extra mortality reserves to be less than the normal, since this constant addition does not allow for the increase in the rates at older ages being sufficiently great to give equal policy-values.

(3) The difference between the extra rates of mortality and the normal may be great at first, afterwards diminishing, and finally disappearing, until the two tables coincide. Here also the reserves

under the table of extra mortality will be smaller than the normal, as the increase is proportionately greater at the younger ages.

The effect upon policy-values of an increase in the mortality under a table graduated by Makeham's formula, $\mu_x = A + Bc^x$, may be considered briefly.

If in this formula the value of A be increased, the result, as shown on page 232, is equivalent to increasing the rate of interest, and that, as proved in *Text Book*, Articles 69 and 70, results in a lower policy-value. A lower policy-value is therefore the effect of a constant addition to the force of mortality. If, on the other hand, B be increased, the effect, as shown on page 233, is to increase the age, and therefore a higher policy-value will be given. If A and B be both increased, the ultimate effect cannot be ascertained without further investigation, as the two increases operate in opposite directions in their effect on policy-values.

There are two well-known methods of dealing with extra-rated cases in valuation.

(1) Policies on such lives may be valued at the increased ages which correspond to the higher rates of premium charged; that is, they are treated throughout as normal policies effected at such increased ages.

(2) The policies may be valued at the true age, precisely like normal policies of that age, each year's extra premiums being assumed to meet that year's extra claims.

It is interesting to examine the extra rates of mortality which are assumed to underlie each of these methods.

Suppose (x) to be a life charged the premium as at age $x+r$. Under the first method we then have

Years Elapsed.	Normal Rate of Mortality.	Assumed Rate of Mortality.	Extra Rate of Mortality.
0	q_x	q_{x+r}	$q_{x+r} - q_x$
1	q_{x+1}	q_{x+r+1}	$q_{x+r+1} - q_{x+1}$
2	q_{x+2}	q_{x+r+2}	$q_{x+r+2} - q_{x+2}$
⋮	⋮	⋮	⋮
t	q_{x+t}	q_{x+r+t}	$q_{x+r+t} - q_{x+t}$

In examining the second method, we know that for a normal life

$$({}_nV_x + P_x)(1+i) = {}_{n+1}V_x + q_{x+n}(1 - {}_{n+1}V_x)$$

while in the case of the extra-rated life under this method

$$({}_nV_x + P_x + R)(1+i) = {}_{n+1}V_x + q'_{x+n}(1 - {}_{n+1}V_x)$$

where R is the extra premium, and q'_{x+n} represents the actual rate of mortality. That is, the normal reserve plus the ordinary net premium and extra premium accumulated to the end of the year are assumed to be sufficient to meet the normal reserve at the end of the year, and make the necessary contribution towards payment of the claims actually experienced. Hence

$$R(1+i) = (q'_{x+n} - q_{x+n})(1 - {}_{n+1}V_x)$$

and
$$(q'_{x+n} - q_{x+n}) = \frac{R(1+i)}{1 - {}_{n+1}V_x}$$
$$= \frac{R(1+i)a_x}{a_{x+n+1}}$$

where $(q'_{x+n} - q_{x+n})$ represents the extra rate of mortality which it is desired to ascertain. Since $R = P_{x+r} - P_x$ we have

Years Elapsed.	Extra Rate of Mortality.
0	$\frac{1}{a_{x+1}}(P_{x+r} - P_x)(1+i)a_x$
1	$\frac{1}{a_{x+2}}(P_{x+r} - P_x)(1+i)a_x$
2	$\frac{1}{a_{x+3}}(P_{x+r} - P_x)(1+i)a_x$
⋮	⋮
t	$\frac{1}{a_{x+t+1}}(P_{x+r} - P_x)(1+i)a_x$

12. The proof that a decrease in the rate of interest increases policy-values, and *vice versa*, which is given in *Text Book*, Article

69, is made to depend on the conclusions of *Text Book*, Article 59. Conversely, the proof of *Text Book*, Article 59, may be shown to depend on the proposition that an increase in the rate of interest decreases policy-values as proved in *Text Book*, Article 70.

Thus, let p_x be diminished at each age by a constant percentage.

$$\begin{aligned}\text{Then}\quad a'_x &= vp'_x + v^2p'_xp'_{x+1} + \cdots\\ &= vp_x(1-r) + v^2p_x(1-r)p_{x+1}(1-r) + \cdots\\ &= v'p_x + (v')^2{}_2p_x + \cdots\\ &= a''_x \text{ calculated from the normal mortality table}\end{aligned}$$

at rate of interest j, which is such that

$$v' = \frac{1}{1+j} = \frac{1-r}{1+i};\ \text{whence } \frac{1}{1+j} < \frac{1}{1+i} \text{ and } j > i.$$

$$\begin{aligned}\text{Now}\qquad {}_nV'_x &= 1 - \frac{1+a'_{x+n}}{1+a'_x}\\ &= 1 - \frac{1+a''_{x+n}}{1+a''_x}\\ &= {}_nV''_x \text{ calculated from the normal}\end{aligned}$$

mortality table at rate of interest j.

But since $j > i$, ${}_nV''_x < {}_nV_x$ calculated at rate i, and therefore ${}_nV'_x < {}_nV_x$.

Hence it is seen that the effect of diminishing p_x at each age by a constant percentage is equivalent to using a higher rate of interest; that is, policy-values are diminished.

By a similar process it may be shown that the effect of increasing p_x by a constant percentage is to increase policy-values.

18. The propositions of *Text Book*, Articles 71 and 72, may be proved as follows:—

If $a_x < a_{x+1}$

$$\text{then}\qquad vp_x(1+a_{x+1}) < a_{x+1}$$

$$\text{and}\qquad vp_x < \frac{a_{x+1}}{1+a_{x+1}}$$

$$v - vp_x > v - \frac{a_{x+1}}{1+a_{x+1}}$$

$$\text{that is,}\qquad P^1_{x:\overline{1}|} > P_{x+1}$$

Again, if $a_x < a_{x+1}$

$$vp_x(1+a_x) < vp_x(1+a_{x+1})$$
$$< a_x$$
$$vp_x < \frac{a_x}{1+a_x}$$
$$v - vp_x > v - \frac{a_x}{1+a_x}$$

that is, $$P^{1}_{x:\overline{1}|} > P_x$$

Further, if $a_x < a_{x+1}$

then $$\frac{1}{1+a_x} - d > \frac{1}{1+a_{x+1}} - d$$

that is $$P_x > P_{x+1}$$

Finally, $${}_1V_x = \frac{a_x - a_{x+1}}{1+a_x}$$

Therefore, if $a_x < a_{x+1}$, ${}_1V_x$ is negative.

14. Besides the two cases mentioned in *Text Book*, Article 73, in which negative policy-values occur, the following may also be noticed :—

(1) Reversionary Annuity Contracts.

Here we have

$${}_nVa_{y|x} = a_{y+n|x+n} - Pa_{y|x}\mathrm{a}_{x+n:y+n}$$
$$= \left(\frac{a_{x+n} - a_{x+n:y+n}}{\mathrm{a}_{x+n:y+n}} - \frac{a_x - a_{xy}}{\mathrm{a}_{xy}}\right)\mathrm{a}_{x+n:y+n}$$

It has already been pointed out (page 274) that the annual premium for such a policy may decrease with an increase in the ages. If this were to happen, the above formula would give a negative result.

(2) Contingent Insurance, $-(x)$ against the survivor of (y) and (z), (z) having died.

Under such conditions

$${}_nV^{1}_{x:\overline{yz}} = A^{\;1}_{x+n:\overline{y+n}} - P^{1}_{x:\overline{yz}}\mathrm{a}_{x+n:y+n}$$
$$= (P^{\;1}_{x+n:\overline{y+n}} - P^{1}_{x:\overline{yz}})\mathrm{a}_{x+n:y+n}$$

As pointed out on page 247, if n be small, $P_{\frac{1}{x+n}:\overline{y+n}}$ may be less than $P^1_{x:\overline{yr}}$, in which case this value will be negative.

15. To prove $\quad {}_nV_x^{(m)} = {}_nV_x\left(1+\frac{m-1}{2m}P_x^{(m)}\right)$

$$\begin{aligned}
{}_nV_x^{(m)} &= A_{x+n} - P_x^{(m)}a_{x+n}^{(m)}\\
&= A_{x+n} - P_x^{(m)}\left(a_{x+n}-\frac{m-1}{2m}\right)\\
&= A_{x+n} - \left\{P_x+\frac{m-1}{2m}P_x^{(m)}(P_x+d)\right\}a_{x+n}+\frac{m-1}{2m}P_x^{(m)}
\end{aligned}$$

(since $P_x^{(m)} = P_x+\frac{m-1}{2m}P_x^{(m)}(P_x+d)$, see page 196)

$$\begin{aligned}
&= A_{x+n} - P_x a_{x+n} + \frac{m-1}{2m}P_x^{(m)}\{1-(P_x+d)a_{x+n}\}\\
&= {}_nV_x\left(1+\frac{m-1}{2m}P_x^{(m)}\right)
\end{aligned}$$

Again

$$\begin{aligned}
{}_nV_{\overline{xr}|}^{(m)} &= A_{x+n:\overline{r-n}|} - P_{\overline{xr}|}^{(m)}a_{x+n:\overline{r-n}|}^{(m)}\\
&= A_{x+n:\overline{r-n}|} - P_{\overline{xr}|}^{(m)}\left\{a_{x+n:\overline{r-n}|}-\frac{m-1}{2m}\left(1-\frac{D_{x+r}}{D_{x+n}}\right)\right\}\\
&= A_{x+n:\overline{r-n}|} - \left\{P_{\overline{xr}|}+\frac{m-1}{2m}P_{\overline{xr}|}^{(m)}(P^1_{\overline{xr}|}+d)\right\}a_{x+n:\overline{r-n}|}+\frac{m-1}{2m}P_{\overline{xr}|}^{(m)}\left(1-\frac{D_{x+r}}{D_{x+n}}\right)
\end{aligned}$$

(since $P_{\overline{xr}|}^{(m)} = P_{\overline{xr}|}+\frac{m-1}{2m}P_{\overline{xr}|}^{(m)}(P^1_{\overline{xr}|}+d)$, see page 198).

$$\begin{aligned}
&= A_{x+n:\overline{r-n}|} - P_{\overline{xr}|}a_{x+n:\overline{r-n}|}+\frac{m-1}{2m}P_{\overline{xr}|}^{(m)}\left\{1-\frac{D_{x+r}}{D_{x+n}}-(P^1_{\overline{xr}|}+d)a_{x+n:\overline{r-n}|}\right\}\\
&= {}_nV_{\overline{xr}|}+\frac{m-1}{2m}P_{\overline{xr}|}^{(m)}\,{}_nV^1_{\overline{xr}|}
\end{aligned}$$

Or, better

$$\begin{aligned}
{}_nV_{\overline{xr}|}^{(m)} &= A_{x+n:\overline{r-n}|} - P_{\overline{xr}|}^{(m)}a_{x+n:\overline{r-n}|}^{(m)}\\
&= A_{x+n:\overline{r-n}|} - P_{\overline{xr}|}^{(m)}\left\{a_{x+n:\overline{r-n}|}-\frac{m-1}{2m}(P_{\frac{1}{x+n}:\overline{r-n}|}+d)a_{x+n:\overline{r-n}|}\right\}
\end{aligned}$$

(substituting for $a^{(m)}_{x+n:\overline{r-n|}}$ its value as found on page 198)

$$= A_{x+n:\overline{r-n|}} - \left\{P_{\overline{xr|}} + \frac{m-1}{2m}P^{(m)}_{\overline{xr|}}(P^1_{\overline{xr|}} + d)\right\}a_{x+n:\overline{r-n|}}$$

$$+ \frac{m-1}{2m}P^{(m)}_{\overline{xr|}}(P^{\;1}_{x+n:\overline{r-n|}} + d)a_{x+n:\overline{r-n|}}$$

$$= A_{x+n:\overline{r-n|}} - P_{\overline{xr|}}a_{x+n:\overline{r-n|}} + \frac{m-1}{2m}P^{(m)}_{\overline{xr|}}(P^{\;1}_{x+n:\overline{r-n|}} - P^1_{\overline{xr|}})a_{x+n:\overline{r-n|}}$$

$$= {}_nV_{\overline{xr|}} + \frac{m-1}{2m}P^{(m)}_{\overline{xr|}}\,{}_nV^1_{\overline{xr|}}$$

Similarly, $${}_{n:r}V^{(m)}_x = {}_{n:r}V_x + \frac{m-1}{2m}\,{}_rP^{(m)}_x\,{}_nV^1_{\overline{xr|}}$$

16. To find ${}_{n+t}V_x$ (n being an integer and t a fraction).

$${}_{n+t}V_x = A_{x+n+t} - P_x \times {}_{1-t|}a_{x+n+t}$$

Now $${}_{0|}a_y = a_y - 0$$

and $${}_{1|}a_y = a_y - 1$$

Therefore, interpolating by first differences, where κ is any fraction of a year

$${}_{\kappa|}a_y = a_y - \kappa$$

Hence $${}_{1-t|}a_{x+n+t} = a_{x+n+t} - (1-t)$$

and $${}_{n+t}V_x = A_{x+n+t} - P_x a_{x+n+t} + P_x(1-t)$$

which agrees with *Text Book* formula (31).

17. To find ${}_{n+t}V^{(m)}_x$, $\left(t = \frac{k}{m}\right)$

$${}_{n+t}V^{(m)}_x = A_{x+n+t} - P^{(m)}_x a^{(m)}_{x+n+t}$$

$$= A_{x+n+t} - \left\{P_x + \frac{m-1}{2m}P^{(m)}_x(P_x + d)\right\}a_{x+n+t} + P^{(m)}_x\frac{m-1}{2m}$$

$$= A_{x+n+t} - P_x a_{x+n+t} + \frac{m-1}{2m}P^{(m)}_x\{1 - (P_x + d)a_{x+n+t}\}$$

$$= (A_{x+n+t} - P_x a_{x+n+t})\left(1 + \frac{m-1}{2m}P^{(m)}_x\right)$$

which agrees with *Text Book* formula (36), since

$$
\begin{aligned}
{}_nV_x + t({}_{n+1}V_x - {}_nV_x) &= A_{x+n} - P_x a_{x+n} + t(A_{x+n+1} - P_x a_{x+n+1}) \\
&\quad - t(A_{x+n} - P_x a_{x+n}) \\
&= A_{x+n} + t(A_{x+n+1} - A_{x+n}) \\
&\quad - P_x\{a_{x+n} + t(a_{x+n+1} - a_{x+n})\} \\
&= A_{x+n+t} - P_x a_{x+n+t}
\end{aligned}
$$

if first differences are taken to be constant.

To find ${}_{n+t}V_x^{(m)}$, $\left(t = \dfrac{k}{m} + s\right)$

$$
\begin{aligned}
{}_{n+t}V_x^{(m)} &= A_{x+n+t} - P_x^{(m)} \times {}_{\frac{1}{m}-s}|a^{(m)}_{x+n+t} \\
&= A_{x+n+t} - P_x^{(m)}\left(a^{(m)}_{x+n+t} - \frac{1}{m} + s\right) \\
&= A_{x+n+t} - P_x^{(m)} a^{(m)}_{x+n+t} + P_x^{(m)}\left(\frac{1}{m} - s\right) \\
&= A_{x+n+t} - P_x^{(m)}\left(a_{x+n+t} - \frac{m-1}{2m}\right) + P_x^{(m)}\left(\frac{1}{m} - s\right)
\end{aligned}
$$

$$
\begin{aligned}
= A_{x+n+t} &- \left\{P_x + \frac{m-1}{2m} P_x^{(m)}(P_x + d)\right\} a_{x+n+t} + \frac{m-1}{2m} P_x^{(m)} \\
&+ \left\{P_x + \frac{m-1}{2m} P_x^{(m)}(P_x + d)\right\}\left\{(1-t) - \left(1 - \frac{k+1}{m}\right)\right\}
\end{aligned}
$$

since $\quad \dfrac{1}{m} - s = \dfrac{k+1}{m} - \left(\dfrac{k}{m} + s\right) = (1-t) - \left(1 - \dfrac{k+1}{m}\right)$

$$
\begin{aligned}
\text{Hence } {}_{n+t}V_x^{(m)} &= A_{x+n+t} - P_x a_{x+n+t} + P_x(1-t) - P_x\left(1 - \frac{k+1}{m}\right) \\
&+ \frac{m-1}{2m} P_x^{(m)}\left\{1 - (P_x + d)a_{x+n+t} + P_x(1-t) - P_x\left(1 - \frac{k+1}{m}\right)\right\} \\
&\quad + d\,\frac{m-1}{2m} P_x^{(m)}\left(\frac{1}{m} - s\right) \\
= {}_{n+t}V_x &- P_x\left(1 - \frac{k+1}{m}\right) + \frac{m-1}{2m} P_x^{(m)}\left\{{}_{n+t}V_x - P_x\left(1 - \frac{k+1}{m}\right)\right\} \\
&\quad + d\,\frac{m-1}{2m} P_x^{(m)}\left(\frac{1}{m} - s\right)
\end{aligned}
$$

The third and fourth terms of this expression are very small, and in practice are usually ignored, it being assumed that

$$\begin{aligned}{}_{n+t}V_x^{(m)} &= {}_{n+t}V_x - P_x\left(1 - \frac{k+1}{m}\right)\\ &= A_{x+n+t} - P_x a_{x+n+t} + P_x\left(\frac{1}{m} - s\right)\end{aligned}$$

It may be noted that where the $m$thly premiums are instalment premiums, the third term will not appear in the expression for the exact value of ${}_{n+t}V_x^{[m]}$, and therefore the value used in practice will in that case be so much nearer the true value.

To obtain ${}_{n+t}V_x^{(m)}\left(t = \frac{k}{m} + s\right)$, we may also proceed by interpolating between

$$\left({}_{n+\frac{k}{m}}V_x^{(m)} + \frac{P_x^{(m)}}{m}\right) \text{ and } {}_{n+\frac{k+1}{m}}V_x^{(m)}$$

Thus

$$\begin{aligned}{}_{n+t}V_x^{(m)} &= {}_{n+\frac{k}{m}}V_x^{(m)} + \frac{P_x^{(m)}}{m} + sm\left({}_{n+\frac{k+1}{m}}V_x^{(m)} - {}_{n+\frac{k}{m}}V_x^{(m)} - \frac{P_x^{(m)}}{m}\right)\\ &= {}_{n+\frac{k}{m}}V_x^{(m)} + sm\left({}_{n+\frac{k+1}{m}}V_x^{(m)} - {}_{n+\frac{k}{m}}V_x^{(m)}\right) + P_x^{(m)}\left(\frac{1}{m} - s\right)\end{aligned}$$

Now, substituting for ${}_{n+\frac{k}{m}}V_x^{(m)}$ and ${}_{n+\frac{k+1}{m}}V_x^{(m)}$ their values as found by *Text Book* formula (36), we have

$$\begin{aligned}{}_{n+t}V_x^{(m)} &= \left(1 + \frac{m-1}{2m}P_x^{(m)}\right)\Big[{}_nV_x + \frac{k}{m}({}_{n+1}V_x - {}_nV_x)\\ &\quad + sm\left\{{}_nV_x + \frac{k+1}{m}({}_{n+1}V_x - {}_nV_x) - {}_nV_x - \frac{k}{m}({}_{n+1}V_x - {}_nV_x)\right\}\Big] + P_x^{(m)}\left(\frac{1}{m} - s\right)\\ &= \left(1 + \frac{m-1}{2m}P_x^{(m)}\right)\left\{{}_nV_x + \left(\frac{k}{m} + s\right)({}_{n+1}V_x - {}_nV_x)\right\} + P_x^{(m)}\left(\frac{1}{m} - s\right)\end{aligned}$$

The quantity $\frac{m-1}{2m}P_x^{(m)}$ is very small, and may be ignored; and P_x may be substituted for $P_x^{(m)}$ in the second term. If these alterations be made the expression will agree with *Text Book* formula (38).

18. To find ${}_{n+t}V_{\overline{xr|}}$

$${}_{n+t}V_{\overline{xr|}} = A_{x+n+t:\overline{r-n-t|}} - P_{\overline{xr|}} \times {}_{1-t|}a_{x+n+t:\overline{r-n-1|}}$$

Now
$$
\begin{aligned}
{}_{0|}a_{\overline{ym|}} &= a_{y:\overline{m+0|}} - 0 \\
{}_{1|}a_{\overline{ym|}} &= a_{y:\overline{m+1|}} - 1
\end{aligned}
$$

Hence interpolating by first differences, where κ is any fraction of a year

$$
{}_{\kappa|}a_{\overline{ym|}} = a_{y:\overline{m+\kappa|}} - \kappa
$$

and
$$
{}_{1-t|}a_{x+n+t:\overline{r-n-1|}} = a_{x+n+t:\overline{r-n-t|}} - (1 - t)
$$

$$
{}_{n+t}V_{\overline{xr|}} = A_{x+n+t:\overline{r-n-t|}} - P_{\overline{xr|}}a_{x+n+t:\overline{r-n-t|}} + P_{\overline{xr|}}(1-t)
$$

Or following the method of *Text Book*, Article 78, we have

$$
{}_{n+t}V_{\overline{xr|}} = {}_{n}V_{\overline{xr|}} + t({}_{n+1}V_{\overline{xr|}} - {}_{n}V_{\overline{xr|}}) + P_{\overline{xr|}}(1-t)
$$

The two formulas are identical, since

$$
\begin{aligned}
&{}_{n}V_{\overline{xr|}} + t({}_{n+1}V_{\overline{xr|}} - {}_{n}V_{\overline{xr|}}) \\
&= A_{x+n:\overline{r-n|}} - P_{\overline{xr|}}a_{x+n:\overline{r-n|}} + t\{(A_{x+n+1:\overline{r-n-1|}} - P_{\overline{xr|}}a_{x+n+1:\overline{r-n-1|}}) \\
&\qquad - (A_{x+n:\overline{r-n|}} - P_{\overline{xr|}}a_{x+n:\overline{r-n|}})\} \\
&= A_{x+n:\overline{r-n|}} + t(A_{x+n+1:\overline{r-n-1|}} - A_{x+n:\overline{r-n|}}) \\
&\qquad - P_{\overline{xr|}}\{a_{x+n:\overline{r-n|}} + t(a_{x+n+1:\overline{r-n-1|}} - a_{x+n:\overline{r-n|}})\} \\
&= A_{x+n+t:\overline{r-n-t|}} - P_{\overline{xr|}}a_{x+n+t:\overline{r-n-t|}}
\end{aligned}
$$

19. To find ${}_{n+t}V^{(m)}_{\overline{xr|}}$, $\left(t = \frac{k}{m} + s\right)$

$$
\begin{aligned}
{}_{n+t}V^{(m)}_{\overline{xr|}} &= A_{x+n+t:\overline{r-n-t|}} - P^{(m)}_{\overline{xr|}} \times {}_{\frac{1}{m}-s|}a^{(m)}_{x+n+t:\overline{r-n-\frac{k+1}{m}|}} \\
&= A_{x+n+t:\overline{r-n-t|}} - P^{(m)}_{\overline{xr|}}\left(a^{(m)}_{x+n+t:\overline{r-n-t|}} - \frac{1}{m} + s\right)
\end{aligned}
$$

$$
= A_{x+n+t:\overline{r-n-t|}} - P^{(m)}_{\overline{xr|}}\left\{a_{x+n+t:\overline{r-n-t|}} - a_{x+n+t:\overline{r-n-t|}}(P_{\frac{1}{x+n+t:\overline{r-n-t|}}} + d)\frac{m-1}{2m}\right\}
$$

$$
+ P^{(m)}_{\overline{xr|}}\left(\frac{1}{m} - s\right)
$$

$$
{}_{n+t}V^{(m)}_{\overline{xr}|} = A_{x+n+t:\overline{r-n-t}|} - \left\{P_{\overline{xr}|} + \frac{m-1}{2m}P^{(m)}_{\overline{xr}|}(P^{1}_{\overline{xr}|}+d)\right\}a_{x+n+t:\overline{r-n-t}|}
$$

$$
+\frac{m-1}{2m}P^{(m)}_{\overline{xr}|}(P_{\overset{1}{x+n+t}:\overline{r-n-t}|}+d)a_{x+n+t:\overline{r-n-t}|}
$$

$$
+\left\{P_{\overline{xr}|} + \frac{m-1}{2m}P^{(m)}_{\overline{xr}|}(P^{1}_{\overline{xr}|}+d)\right\}\left\{(1-t)-\left(1-\frac{k+1}{m}\right)\right\}
$$

$$
= A_{x+n+t:\overline{r-n-t}|} - P_{\overline{xr}|}a_{x+n+t:\overline{r-n-t}|} + P_{\overline{xr}|}(1-t) - P_{\overline{xr}|}\left(1-\frac{k+1}{m}\right)
$$

$$
+\frac{m-1}{2m}P^{(m)}_{\overline{xr}|}\left\{(P_{\overset{1}{x+n+t}:\overline{r-n-t}|} - P^{1}_{\overline{xr}|})a_{x+n+t:\overline{r-n-t}|} + P^{1}_{\overline{xr}|}(1-t) - P^{1}_{\overline{xr}|}\left(1-\frac{k+1}{m}\right)\right\}
$$

$$
+d\frac{m-1}{2m}P^{(m)}_{\overline{xr}|}\left(\frac{1}{m}-s\right)
$$

$$
= {}_{n+t}V_{\overline{xr}|} - P_{\overline{xr}|}\left(1-\frac{k+1}{m}\right) + \frac{m-1}{2m}P^{(m)}_{\overline{xr}|}\left\{{}_{n+t}V^{1}_{\overline{xr}|} - P^{1}_{\overline{xr}|}\left(1-\frac{k+1}{m}\right)\right\}
$$

$$
+d\frac{m-1}{2m}P^{(m)}_{\overline{xr}|}\left(\frac{1}{m}-s\right)
$$

The third and fourth terms of this expression are very small, and in practice it may be taken that

$$
{}_{n+t}V^{(m)}_{\overline{xr}|} = {}_{n+t}V_{\overline{xr}|} - P_{\overline{xr}|}\left(1-\frac{k+1}{m}\right)
$$

$$
= A_{x+n+t:\overline{r-n-t}|} - P_{\overline{xr}|}a_{x+n+t:\overline{r-n-t}|} + P_{\overline{xr}|}\left(\frac{1}{m}-s\right)
$$

As was the case for whole-life assurances, the third term in the exact expression entirely disappears when the premiums receivable are instalment premiums, and accordingly the approximate expression is in these circumstances so much nearer exactitude.

20. To find ${}_{n+t:r}V_x$

$$
{}_{n+t:r}V_x = A_{x+n+t} - {}_rP_x \times {}_{1-t|}a_{x+n+t:\overline{r-n-1}|}
$$

$$
= A_{x+n+t} - {}_rP_x a_{x+n+t:\overline{r-n-t}|} + {}_rP_x(1-t)
$$

21. To find ${}_{n+t:r}V^{(m)}_x$, $\left(t = \frac{k}{m} + s\right)$

$$
\begin{aligned}
{}_{n+t:r}V^{(m)}_x &= A_{x+n+t} - {}_rP^{(m)}_x\left(a^{(m)}_{x+n+t:\overline{r-n-t}|} - \frac{1}{m} + s\right)\\
&= A_{x+n+t} - \left\{{}_rP_x + \frac{m-1}{2m}\,{}_rP^{(m)}_x(P^{1}_{\overline{xr}|} + d)\right\}a_{x+n+t:\overline{r-n-t}|}\\
&\quad + \frac{m-1}{2m}\,{}_rP^{(m)}_x(P_{\overline{x+n+t:r-n-t}|}^{\;1} + d)a_{x+n+t:\overline{r-n-t}|}\\
&\quad + \left\{{}_rP_x + \frac{m-1}{2m}\,{}_rP^{(m)}_x(P^{1}_{\overline{xr}|} + d)\right\}\left\{(1-t) - \left(1 - \frac{k+1}{m}\right)\right\}\\
&= A_{x+n+t} - {}_rP_x a_{x+n+t:\overline{r-n-t}|} + {}_rP_x(1-t) - {}_rP_x\left(1 - \frac{k+1}{m}\right)\\
&\quad + \frac{m-1}{2m}\,{}_rP^{(m)}_x\left\{(P_{\overline{x+n+t:r-n-t}|}^{\;1} - P^{1}_{\overline{xr}|})a_{x+n+t:\overline{r-n-t}|} + P^{1}_{\overline{xr}|}(1-t) - P^{1}_{\overline{xr}|}\left(1 - \frac{k+1}{m}\right)\right\}\\
&\quad + d\,\frac{m-1}{2m}\,{}_rP^{(m)}_x\left(\frac{1}{m} - s\right)\\
&= {}_{n+t:r}V_x - {}_rP_x\left(1 - \frac{k+1}{m}\right) + \frac{m-1}{2m}\,{}_rP^{(m)}_x\left\{{}_{n+t}V^{1}_{\overline{xr}|} - P^{1}_{\overline{xr}|}\left(1 - \frac{k+1}{m}\right)\right\}\\
&\quad + d\,\frac{m-1}{2m}\,{}_rP^{(m)}_x\left(\frac{1}{m} - s\right)
\end{aligned}
$$

The third and fourth terms are again very small and may be ignored, and we shall have

$$
\begin{aligned}
{}_{n+t:r}V^{(m)}_x &= {}_{n+t:r}V_x - {}_rP_x\left(1 - \frac{k+1}{m}\right)\\
&= A_{x+n+t} - {}_rP_x a_{x+n+t:\overline{r-n-t}|} + {}_rP_x\left(\frac{1}{m} - s\right)
\end{aligned}
$$

22. In practice, however, these expressions for limited-payment policies are inadmissible, as explained in *Text Book*, Articles 115 to 119; and we may modify *Text Book* formula (55) as follows:—

$$
\begin{aligned}
{}_{n+t:r}U_x = A_{x+n+t} &- {}_rP'_x(1-c)_{1-t|}a_{x+n+t:\overline{r-n-1}|}\\
&+ \{P'_x(1-c) - P_x\}_{1-t|}a_{x+n+t}
\end{aligned}
$$

and $${}_{n+t:r}U^{(m)}_x = {}_{n+t:r}U_x - {}_rP'_x(1-c)\left(1 - \frac{k+1}{m}\right)$$

The formulas given in sections 15 to 22 were deduced by Mr J. J. M'Lauchlan (*Trans. Act. Soc. Edin.*, Vol. II., No. 12).

23. To find the reserve after n years of a pure endowment payable at age $(x+t)$, with return of premiums in the event of previous death.

By the prospective method

$$
{}_nV = \frac{D_{x+t}}{D_{x+n}} + \{\pi(1+\kappa)+c\}\frac{nM_{x+n} + R_{x+n} - R_{x+t} - tM_{x+t}}{D_{x+n}}
$$

$$
-\pi\frac{N_{x+n-1} - N_{x+t-1}}{D_{x+n}}
$$

By the retrospective method, the premiums returned at the end of each year accumulated to the end of n years, for l_x policies taken out, amount to

$$
\{\pi(1+\kappa)+c\}\{d_x(1+i)^{n-1} + 2d_{x+1}(1+i)^{n-2} + 3d_{x+2}(1+i)^{n-3} + \cdots
$$

$$
+ nd_{x+n-1}\}
$$

$$
= \{\pi(1+\kappa)+c\}\frac{C_x + 2C_{x+1} + 3C_{x+2} + \ldots + nC_{x+n-1}}{v^{x+n}}
$$

$$
= \{\pi(1+\kappa)+c\}\frac{R_x - R_{x+n} - nM_{x+n}}{v^{x+n}}
$$

Dividing this result by the number of survivors after n years we have

$$
\{\pi(1+\kappa)+c\}\frac{R_x - R_{x+n} - nM_{x+n}}{D_{x+n}}
$$

The reserve by the retrospective method is therefore

$$
{}_nV = \frac{\pi(N_{x-1} - N_{x+n-1}) - \{\pi(1+\kappa)+c\}(R_x - R_{x+n} - nM_{x+n})}{D_{x+n}}
$$

The two expressions are identical, for

$$\frac{\pi(N_{x-1}-N_{x+n-1})-\{\pi(1+\kappa)+c\}(R_x-R_{x+n}-nM_{x+n})}{D_{x+n}}$$

$$\begin{aligned}
&= \frac{1}{D_{x+n}}\Big[\pi\{N_{x-1}-N_{x+t-1}-(1+\kappa)(R_x-R_{x+t}-tM_{x+t})\} \\
&\qquad -\{D_{x+t}+c(R_x-R_{x+t}-tM_{x+t})\} \\
&\qquad +D_{x+t}+\{\pi(1+\kappa)+c\}(nM_{x+n}+R_{x+n}-R_{x+t}-tM_{x+t}) \\
&\qquad -\pi(N_{x+n-1}-N_{x+t-1})\Big] \\
&= \frac{D_{x+t}}{D_{x+n}}+\{\pi(1+\kappa)+c\}\frac{nM_{x+n}+R_{x+n}-R_{x+t}-tM_{x+t}}{D_{x+n}} \\
&\qquad -\pi\frac{N_{x+n-1}-N_{x+t-1}}{D_{x+n}}
\end{aligned}$$

since $\pi = \dfrac{D_{x+t}+c(R_x-R_{x+t}-tM_{x+t})}{N_{x-1}-N_{x+t-1}-(1+\kappa)(R_x-R_{x+t}-tM_{x+t})}$

As explained on page 318 the premium for this benefit is usually calculated without taking any account of the element of mortality. The premium charged would therefore be

$$P_{\overline{t|}} = \frac{1}{(1+i)s_{\overline{t|}}} \text{ or } \frac{v^t}{a_{\overline{t|}}}$$

with a suitable loading for expenses.

The reserve accordingly might be taken as the net premiums accumulated at rate i to the date of valuation.

$$P_{\overline{t|}}(1+i)s_{\overline{n|}} = \frac{s_{\overline{n|}}}{s_{\overline{t|}}}$$

24. To find the reserve under a similar policy with the addition that simple interest at rate j is to be returned with the premiums in event of death before age $(x+t)$.

Prospectively, the reserve is

$$\frac{D_{x+t}}{D_{x+n}} + \{\pi(1+\kappa)+c\}\frac{nM_{x+n}+R_{x+n}-R_{x+t}-tM_{x+t}}{D_{x+n}}$$

$$+j\{\pi(1+\kappa)+c\}\frac{\frac{n(n+1)}{2}M_{x+n}+nR_{x+n}+\Sigma R_{x+n}-\Sigma R_{x+t}-tR_{x+t}-\frac{t(t+1)}{2}M_{x+t}}{D_{x+n}}$$

$$-\pi\frac{N_{x+n-1}-N_{x+t-1}}{D_{x+n}}$$

The third term here is made up of the liability for return of interest on past premiums and that on future premiums. The former is equal to

$$j\pi'\frac{\frac{n(n+1)}{2}(M_{x+n}-M_{x+t})+n\{R_{x+n}-R_{x+t}-(t-n)M_{x+t}\}}{D_{x+n}}$$

$$=j\pi'\frac{\frac{n(n+1)}{2}M_{x+n}+n(R_{x+n}-R_{x+t})-\frac{n(2t-n+1)}{2}M_{x+t}}{D_{x+n}}$$

And the latter is equal to

$$j\pi'\frac{\Sigma R_{x+n}-\Sigma R_{x+t}-(t-n)R_{x+t}-\frac{(t-n)(t-n+1)}{2}M_{x+t}}{D_{x+n}}$$

Together, as above,

$$j\pi'\frac{\frac{n(n+1)}{2}M_{x+n}+nR_{x+n}+\Sigma R_{x+n}-\Sigma R_{x+t}-tR_{x+t}-\frac{t(t+1)}{2}M_{x+t}}{D_{x+n}}$$

The reserve found retrospectively is

$$\frac{1}{D_{x+n}}\Big[\pi(N_{x-1}-N_{x+n-1})-\{\pi(1+\kappa)+c\}(R_x-R_{x+n}-nM_{x+n})$$
$$-j\{\pi(1+\kappa)+c\}\left\{\Sigma R_x-\Sigma R_{x+n}-nR_{x+n}-\frac{n(n+1)}{2}M_{x+n}\right\}\Big].$$

The two formulas may easily be proved to be identical if the value of π as found on page 309 be remembered.

25. To find the reserve under a similar policy except that the premiums are to be returned with compound interest at rate j in the event of death before age $(x+t)$.

Prospectively, the reserve is

$$\frac{D_{x+t}}{D_{x+n}} + \{\pi(1+\kappa)+c\}\frac{1}{D_{x+n}}\{(1+j)s_{\overline{n}|}(M_{x+n} - M_{x+t}) + (1+j)^{n+1}(M_{x+n} - M_{x+t})$$
$$+ (1+j)^{n+2}(M_{x+n+1} - M_{x+t}) + \cdot \cdot \cdot + (1+j)^{t}(M_{x+t-1} - M_{x+t})\}$$
$$- \pi(1 + a_{x+n:\overline{t-n-1}|})$$

And retrospectively

$$\pi\frac{N_{x-1} - N_{x+n-1}}{D_{x+n}} - \{\pi(1+\kappa)+c\}\frac{1}{D_{x+n}}\{(1+j)(M_{x} - M_{x+n})$$
$$+ (1+j)^{2}(M_{x+1} - M_{x+n}) + \cdot \cdot \cdot + (1+j)^{n}(M_{x+n-1} - M_{x+n})\}$$

These two should be proved equal, given the value of π as found on page 310.

26. To find the value after n years of an assurance deferred t years, premiums payable throughout life but returnable in the event of death within t years.

Three cases arise, viz., $n < = > t$.

(1) $n < t$.

Prospectively, the reserve is

$$\frac{M_{x+t}}{D_{x+n}} + \{\pi(1+\kappa)+c\}\frac{nM_{x+n} + R_{x+n} - R_{x+t} - tM_{x+t}}{D_{x+n}} - \frac{\pi N_{x+n-1}}{D_{x+n}}$$

Retrospectively

$$\frac{\pi(N_{x-1} - N_{x+n-1})}{D_{x+n}} - \{\pi(1+\kappa)+c\}\frac{R_{x} - R_{x+n} - nM_{x+n}}{D_{x+n}}$$

Now these two expressions are identical, for

$$\frac{1}{D_{x+n}}[\pi(N_{x-1} - N_{x+n-1}) - \{\pi(1+\kappa)+c\}(R_{x} - R_{x+n} - nM_{x+n})]$$

$$= \frac{1}{D_{x+n}}[\pi\{N_{x-1} - (1+\kappa)(R_{x} - R_{x+t} - tM_{x+t})\} - M_{x+t} - c(R_{x} - R_{x+t} - tM_{x+t})$$
$$+ M_{x+t} + \pi(1+\kappa)(nM_{x+n} + R_{x+n} - R_{x+t} - tM_{x+t})$$
$$+ c(nM_{x+n} + R_{x+n} - R_{x+t} - tM_{x+t}) - \pi N_{x+n-1}]$$

$$= \frac{1}{D_{x+n}}[M_{x+t} + \{\pi(1+\kappa)+c\}(nM_{x+n} + R_{x+n} - R_{x+t} - tM_{x+t}) - \pi N_{x+n-1}]$$

$$\text{since} \quad \pi = \frac{M_{x+t} + c(R_{x} - R_{x+t} - tM_{x+t})}{N_{x-1} - (1+\kappa)(R_{x} - R_{x+t} - tM_{x+t})}.$$

(2) $n=t$.

Prospectively $\qquad \dfrac{M_{x+t}}{D_{x+t}} - \pi\dfrac{N_{x+t-1}}{D_{x+t}}$

Retrospectively

$$\pi\cdot\frac{N_{x-1}-N_{x+t-1}}{D_{x+t}} - \{\pi(1+\kappa)+c\}\frac{R_x - R_{x+t} - tM_{x+t}}{D_{x+t}}$$

which may easily be proved identical.

(3) $n > t$.

Prospectively $\qquad \dfrac{M_{x+n}}{D_{x+n}} - \pi\dfrac{N_{x+n-1}}{D_{x+n}}$

Retrospectively

$$\pi\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}} - \{\pi(1+\kappa)+c\}\frac{R_x - R_{x+t} - tM_{x+t}}{D_{x+n}} - \frac{M_{x+t}-M_{x+n}}{D_{x+n}}$$

These two expressions also may easily be proved identical.

We have seen that when $n=t$, the reserve which the office must have in hand is $A_{x+t} - \pi a_{x+t}$. This has to be provided out of the premiums received, and the mortality up to age $(x+t)$ may be ignored, as the premiums are returned in the event of death previous to that age. We may therefore find at what rate of interest the premium calculated by the exact formula will amount in t years certain to this amount. That is, find j such that

$$\pi(1+j)s_{\overline{t|}(j)} = A_{x+t} - \pi a_{x+t}$$

The reserve, when $n < t$, may then be taken as $\pi(1+j)s_{\overline{n|}(j)}$.

27. To find the value of a similar deferred assurance, but with the condition that during the t years the premiums shall be paid only so long as another life (y) survives.

(1) When $n < t$.

(*a*) (y) still alive, occurring in ${}_np_y l_{x+n}$ cases out of l_{x+n}.

Prospectively

$$\frac{M_{x+t}}{D_{x+n}} + \{\pi(1+\kappa)+c\}\frac{1}{D_{x+n}}\{(n+1)(M_{x+n}-M_{x+t}) + p_{y+n}(M_{x+n+1}-M_{x+t}) + \cdots$$
$$+ {}_{t-n-1}p_{y+n}(M_{x+t-1}-M_{x+t})\} - \pi\left(\frac{N_{x+n-1:y+n-1} - N_{x+t-1:y+t-1}}{D_{x+n:y+n}} + \frac{N_{x+t-1}}{D_{x+n}}\right)$$

Retrospectively

$$\pi\,\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}-\{\pi(1+\kappa)+c\}\,\frac{R_x-R_{x+n}-nM_{x+n}}{D_{x+n}}$$

(*b*) If (*y*) died after *m* payments had been made, occurring in $({}_{m-1}p_y-{}_mp_y)l_{x+n}$ cases out of l_{x+n}.

Prospectively

$$\frac{M_{x+t}}{D_{x+n}}+m\{\pi(1+\kappa)+c\}\,\frac{M_{x+n}-M_{x+t}}{D_{x+n}}-\pi\,\frac{N_{x+t-1}}{D_{x+n}}$$

Retrospectively

$$\pi\,\frac{N_{x-1}-N_{x+m-1}}{D_{x+n}}-\{\pi(1+\kappa)+c\}\,\frac{R_x-R_{x+m}-mM_{x+n}}{D_{x+n}}$$

(2) When $n=t$.

(*a*) If (*y*) lived $(t-1)$ years, occurring in ${}_{t-1}p_y\,l_{x+t}$ cases out of l_{x+t}.

Prospectively $$\frac{M_{x+t}}{D_{x+t}}-\pi\,\frac{N_{x+t-1}}{D_{x+t}}$$

Retrospectively

$$\pi\,\frac{N_{x-1}-N_{x+t-1}}{D_{x+t}}-\{\pi(1+\kappa)+c\}\,\frac{R_x-R_{x+t}-tM_{x+t}}{D_{x+t}}$$

(*b*) If (*y*) died after *m* $(<t)$ payments, occurring in $({}_{m-1}p_y-{}_mp_y)l_{x+t}$ cases out of l_{x+t}.

Prospectively $$\frac{M_{x+t}}{D_{x+t}}-\pi\,\frac{N_{x+t-1}}{D_{x+t}}$$

Retrospectively

$$\pi\,\frac{N_{x-1}-N_{x+m-1}}{D_{x+t}}-\{\pi(1+\kappa)+c\}\,\frac{R_x-R_{x+m}-mM_{x+t}}{D_{x+t}}$$

(3) When $n>t$.

(*a*) If (*y*) lived $(t-1)$ years, occurring in ${}_{t-1}p_y\,l_{x+n}$ cases out of l_{x+n}.

Prospectively $$\frac{M_{x+n}}{D_{x+n}}-\pi\,\frac{N_{x+n-1}}{D_{x+n}}$$

Retrospectively

$$\pi\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}-\{\pi(1+\kappa)+c\}\frac{R_x-R_{x+t}-tM_{x+t}}{D_{x+n}}-\frac{M_{x+t}-M_{x+n}}{D_{x+n}}$$

(*b*) If (y) died after m ($<t$) payments, occurring in $({}_{m-1}p_y-{}_mp_y)l_{x+n}$ cases out of l_{x+n}.

Prospectively $\qquad \dfrac{M_{x+n}}{D_{x+n}}-\pi\dfrac{N_{x+n-1}}{D_{x+n}}$

Retrospectively

$$\pi\left(\frac{N_{x-1}-N_{x+m-1}}{D_{x+n}}+\frac{N_{x+t-1}-N_{x+n-1}}{D_{x+n}}\right)$$

$$-\{\pi(1+\kappa)+c\}\frac{R_x-R_{x+m}-mM_{x+t}}{D_{x+n}}-\frac{M_{x+t}-M_{x+n}}{D_{x+n}}$$

28. The reserves for limited-payment policies, endowment assurances, and temporary assurances have already been given, both by the retrospective and prospective methods, and the values of each by the two methods are equal.

The three classes may be looked at together in the following manner:—

$$\begin{aligned}
{}_{n:t}V_x &= A_{x+n}-{}_tP_x\mathrm{a}_{x+n:\overline{t-n|}}\\
&= A_{x+n}-{}_tP_x\mathrm{a}_{x+n}+{}_tP_x\frac{N_{x+t-1}}{D_{x+n}}\\
{}_nV_{\overline{xt|}} &= A_{x+n:\overline{t-n|}}-P_{\overline{xt|}}\mathrm{a}_{x+n:\overline{t-n|}}\\
&= A_{x+n}+\frac{dN_{x+t-1}}{D_{x+n}}-P_{\overline{xt|}}\mathrm{a}_{x+n}+P_{\overline{xt|}}\frac{N_{x+t-1}}{D_{x+n}}\\
&= A_{x+n}-P_{\overline{xt|}}\mathrm{a}_{x+n}+\frac{(P_{\overline{xt|}}+d)N_{x+t-1}}{D_{x+n}}\\
{}_nV^1_{\overline{xt|}} &= A^{\ 1}_{x+n:\overline{t-n|}}-P^1_{\overline{xt|}}\mathrm{a}_{x+n:\overline{t-n|}}\\
&= A_{x+n}-\frac{M_{x+t}}{D_{x+n}}-P^1_{\overline{xt|}}\mathrm{a}_{x+n}+P^1_{\overline{xt|}}\frac{N_{x+t-1}}{D_{x+n}}\\
&= A_{x+n}-P^1_{\overline{xt|}}\mathrm{a}_{x+n}+\frac{(P^1_{\overline{xt|}}-P_{x+t})N_{x+t-1}}{D_{x+n}}
\end{aligned}$$

In each of these expressions the first two terms are of the general form $A_{x+n} - Pa_{x+n}$, P alone varying with the nature of the benefit. The third term has $\frac{N_{x+t-1}}{D_{x+n}}$ constant with a varying coefficient. This coefficient consists of two parts, the first of which shows the correction to be made on the second term for the value of the premium, and the second the correction to be made on the first term for the value of the sum assured. Thus in the case of temporary assurances, no premium will be received after age $x+t$, and therefore the value of all premiums after that age must be added to the liability; further, no claims will be paid after that age, and therefore the value of a premium starting then which shall be sufficient to meet all these must be deducted.

29. These formulas may also be worked into the following forms:—

$$\begin{aligned} {}_{n:t}V_x &= A_{x+n} - {}_tP_x a_{x+n} + {}_tP_x \frac{N_{x+t-1}}{D_{x+n}} \\ &= A_{x+n} - {}_tP_x a_{x+n} + \frac{({}_tP_x - P_x)N_{x-1} - {}_tP_x(N_{x-1} - N_{x+t-1}) + P_x N_{x-1}}{D_{x+n}} \\ &= A_{x+n} - {}_tP_x a_{x+n} + \frac{({}_tP_x - P_x)N_{x-1}}{D_{x+n}} \end{aligned}$$

since ${}_tP_x(N_{x-1} - N_{x+t-1}) = M_x = P_x N_{x-1}$

$$\begin{aligned} {}_nV_{\overline{xt}|} &= A_{x+n} - P_{\overline{xt}|} a_{x+n} + \frac{(P_{\overline{xt}|} + d)N_{x+t-1}}{D_{x+n}} \\ &= A_{x+n} - P_{\overline{xt}|} a_{x+n} \\ &\qquad + \frac{(P_{\overline{xt}|} - P_x)N_{x-1} - P_{\overline{xt}|}(N_{x-1} - N_{x+t-1}) + dN_{x+t-1} + P_x N_{x-1}}{D_{x+n}} \\ &= A_{x+n} - P_{\overline{xt}|} a_{x+n} + \frac{(P_{\overline{xt}|} - P_x)N_{x-1}}{D_{x+n}} \end{aligned}$$

since $P_{\overline{xt}|}(N_{x-1} - N_{x+t-1}) - dN_{x+t-1} = M_x - M_{x+t} + D_{x+t} - dN_{x+t-1}$

$$\begin{aligned} &= M_x - vN_{x+t-1} + N_{x+t} + N_{x+t-1} - N_{x+t} - dN_{x+t-1} \\ &= M_x = P_x N_{x-1} \end{aligned}$$

$$
\begin{aligned}
{}_nV^1_{x\overline{t}|} &= A_{x+n} - P^1_{x\overline{t}|}\,a_{x+n} + \frac{(P^1_{x\overline{t}|} - P_{x+t})N_{x+t-1}}{D_{x+n}} \\
&= A_{x+n} - P^1_{x\overline{t}|}\,a_{x+n} \\
&\qquad + \frac{(P^1_{x\overline{t}|} - P_x)N_{x-1} - P^1_{x\overline{t}|}(N_{x-1} - N_{x+t-1}) + P_xN_{x-1} - P_{x+t}N_{x+t-1}}{D_{x+n}} \\
&= A_{x+n} - P^1_{x\overline{t}|}\,a_{x+n} + \frac{(P^1_{x\overline{t}|} - P_x)N_{x-1}}{D_{x+n}}
\end{aligned}
$$

since $P^1_{x\overline{t}|}(N_{x-1} - N_{x+t-1}) = M_x - M_{x+t} = P_xN_{x-1} - P_{x+t}N_{x+t-1}$

Here each expression is of a perfectly general form independent of the value of t, except in so far as t determines the value of the premium for the benefit, $A_{x+n} - Pa_{x+n} + \dfrac{(P - P_x)N_{x-1}}{D_{x+n}}$, where P alone varies, being the premium for the particular benefit under consideration.

30. The reserves for policies under many special schemes may be simply found by remembering the method by which the annual premium was calculated. For example:

(*a*) To find the value after n years of a whole-life policy to (x), under which interest at rate j is to be guaranteed on the sum assured for t years after the death of (x), and thereafter the sum assured is to be payable, the office assuming rate of interest i in its calculations. It will be remembered (see page 148) that the annual premium for this benefit is

$$P_x\{1 + (j - i)a_{\overline{t}|(i)}\}$$

Therefore the reserve will be

$${}_nV_x\{1 + (j - i)a_{\overline{t}|(i)}\}$$

(*b*) To find the value after n years of a whole-life policy, under which the sum assured is to be payable in t equal annual instalments, the first at the end of the year of death of (x). The annual premium for this benefit (see page 147) is

$$P_x \times \frac{a_{\overline{t}|}}{t}$$

and therefore the reserve will be

$$ {}_{n}V_{x} \times \frac{a_{\overline{t}|}}{t} $$

31. To find the value after n years of a policy for the whole of life under which the premium is P for the first t years, and thereafter is 2P.

(1) When $n < t$, the value is

$$ A_{x+n} - P(a_{x+n} + {}_{t-n|}a_{x+n}) \quad \text{Prospectively} $$

$$ \frac{P(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}} \quad \text{Retrospectively} $$

and these two are equal.

(2) When $n =$ or $> t$, the value is

$$ A_{x+n} - 2Pa_{x+n} \quad \text{Prospectively} $$

$$ \frac{P(N_{x-1} + N_{x+t-1} - 2N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}} \quad \text{Retrospectively} $$

which are also equal.

32. To find the value after n years of an endowment assurance policy payable at the end of t years or previous death under which the premium is to be P for the first r years and thereafter 2P.

(1) $n < r$.

Prospectively

$$ A_{x+n:\overline{t-n}|} - P(a_{x+n:\overline{t-n}|} + {}_{r-n|}a_{x+n:\overline{t-r}|}) $$

Retrospectively

$$ \frac{P(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}} $$

(2) $n =$ or $> r$.

Prospectively

$$ A_{x+n:\overline{t-n}|} - 2Pa_{x+n:\overline{t-n}|} $$

Retrospectively

$$ \frac{P(N_{x-1} + N_{x+r-1} - 2N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}} $$

Again in each case the reserves by the two methods may be proved equal.

33. To find the value after n years of a policy subject to a contingent debt should (x) die within t years.

(*a*) Where the debt is X for the whole period of t years.

Assuming that (x) is rated up r years, we have

(1) When $n < t$.

$$\frac{M_{x+r+n}(1-X)+XM_{x+r+t}}{D_{x+r+n}} - \pi_x a_{x+r+n}$$

(2) When $n =$ or $> t$.

$$A_{x+r+n} - \pi_x a_{x+r+n}$$

(*b*) Where the debt is tX for the first year, and decreases by X per annum for t years, we have

(1) When $n < t$.

$$\frac{M_{x+r+n}\{1-(t-n)X\}+X(R_{x+r+n+1}-R_{x+r+t+1})}{D_{x+r+n}} - \pi_x a_{x+r+n}$$

(2) When $n =$ or $> t$.

$$A_{x+r+n} - \pi_x a_{x+r+n}$$

34. To find the value of a double-endowment assurance policy, 2 being payable if (x) live t years, or 1 if he die before that.

$$\begin{aligned}
V &= {}_nV_{\overline{xt}|} + {}_nV_{\overset{1}{x}\overline{t}|} \\
&= {}_nV_{\overline{xt}|} + A_{x+n:\overline{t-n}|}^{\;\;1} - P_{\overset{1}{x}\overline{t}|}\, a_{x+n:\overline{t-n}|} \\
&= {}_nV_{\overline{xt}|} + A_{x+n:\overline{t-n}|}^{\;\;1} - A_{\overset{1}{x}\overline{t}|}\frac{a_{x+n:\overline{t-n}|}}{a_{\overline{xt}|}} \\
&= {}_nV_{\overline{xt}|} + A_{x+n:\overline{t-n}|}^{\;\;1} - A_{\overset{1}{x}\overline{t}|}(1-{}_nV_{\overline{xt}|}) \\
&= {}_nV_{\overline{xt}|}(1+A_{\overset{1}{x}\overline{t}|}) + (A_{x+n:\overline{t-n}|}^{\;\;1} - A_{\overset{1}{x}\overline{t}|})
\end{aligned}$$

Similarly, to find the value of a half-endowment assurance

policy, 1 being payable if (x) live t years, or 2 if he die before that.

$$V = 2\,{}_nV_{\overline{xt}|} - {}_nV_{\overline{xt}|}^{\,1}$$

$$= {}_nV_{\overline{xt}|}(2 - A_{\overline{xt}|}^{\,1}) - (A_{x+n:\overline{t-n}|}^{\,1} - A_{\overline{xt}|}^{\,1})$$

35. To find the value after n years of an annuity deferred years which was purchased by single payment.

(1) $n < t$.

Prospectively $\quad {}_{t-n|}a_{x+n}$

Retrospectively

$$\frac{l_x \times {}_{t|}a_x(1+i)^n}{l_{x+n}} = \frac{{}_{t|}a_x D_x}{D_{x+n}}$$

The two expressions are equal, for

$$\frac{{}_{t|}a_x D_x}{D_{x+n}} = \frac{N_{x+t}}{D_x}\frac{D_x}{D_{x+n}} = \frac{N_{x+t}}{D_{x+n}} = {}_{t-n|}a_{x+n}$$

(2) $n =$ or $> t$.

Prospectively $\quad a_{x+n}$

Retrospectively

$$\frac{{}_{t|}a_x D_x}{D_{x+n}} - \frac{N_{x+t} - N_{x+n}}{D_{x+n}}$$

Again $\quad \dfrac{{}_{t|}a_x D_x - (N_{x+t} - N_{x+n})}{D_{x+n}} = \dfrac{N_{x+n}}{D_{x+n}} = a_{x+n}$

36. To find the value of a similar annuity with the condition that the premium is to be returned if (x) die within the t years.

(1) $n < t$.

Prospectively

$${}_{t-n|}a_{x+n} + \{A(1+\kappa)+c\}A_{x+n:\overline{t-n}|}^{\,1}$$

Retrospectively

$$\frac{AD_x}{D_{x+n}} - \{A(1+\kappa)+c\}\,\frac{M_x - M_{x+n}}{D_{x+n}}$$

But the latter is equal to

$$\frac{1}{D_{x+n}}[A\{D_x-(1+\kappa)(M_x-M_{x+n})\}-c(M_x-M_{x+n})]$$

$$=\frac{1}{D_{x+n}}[A\{D_x-(1+\kappa)(M_x-M_{x+t})\}-\{N_{x+t}+c(M_x-M_{x+t})\}$$
$$+N_{x+t}+\{A(1+\kappa)+c\}(M_{x+n}-M_{x+t})]$$

$$=\frac{N_{x+t}}{D_{x+n}}+\{A(1+\kappa)+c\}\frac{M_{x+n}-M_{x+t}}{D_{x+n}} \text{ since } A=\frac{N_{x+t}+c(M_x-M_{x+t})}{D_x-(1+\kappa)(M_x-M_{x+t})}$$

$$={}_{t-n|}a_{x+n}+\{A(1+\kappa)+c\}A^{\,1}_{x+n:\overline{t-n|}}$$

(2) $n =$ or $> t$.

Prospectively $\quad a_{x+n}$

Retrospectively

$$A\frac{D_x}{D_{x+n}}-\{A(1+\kappa)+c\}\frac{M_x-M_{x+t}}{D_{x+n}}-\frac{N_{x+t}-N_{x+n}}{D_{x+n}}$$

But the latter is equal to

$$\frac{1}{D_{x+n}}[A\{D_x-(1+\kappa)(M_x-M_{x+t})\}-c(M_x-M_{x+t})-(N_{x+t}-N_{x+n})]$$

$$=\frac{1}{D_{x+n}}[A\{D_x-(1+\kappa)(M_x-M_{x+t})\}-\{N_{x+t}+c(M_x-M_{x+t})\}+N_{x+n}]$$

$$=\frac{N_{x+n}}{D_{x+n}}$$

$$=a_{x+n}$$

87. To find the value at the end of n years of an assurance with a uniform reversionary bonus of b per annum declared every five years, an interim bonus of b' being granted in respect of each premium paid since the date of last investigation should the life die within a quinquennium, assuming n to be a multiple of 5.

Whole-Life Policy.

$$\frac{1}{D_{x+n}}[(1+nb)M_{x+n}+5b(M_{x+n+5}+M_{x+n+10}+\cdots)$$
$$+b'\{R_{x+n}-5(M_{x+n+5}+M_{x+n+10}+\cdots)\}]-\pi a_{x+n}$$

π having the value found for the premium for this benefit on page 302.

When $b' = b$, we have

$$\frac{(1+nb)M_{x+n} + bR_{x+n}}{D_{x+n}} - \pi a_{x+n}$$

π having its appropriate value.

Endowment Assurance, maturing at age $(x+t)$.

$$\frac{1}{D_{x+n}}[(1+nb)(M_{x+n} - M_{x+t} + D_{x+t}) + 5b(M_{x+n+5} + M_{x+n+10} + \cdots + M_{x+t})$$
$$+ (t-n)b(D_{x+t} - M_{x+t}) + b'\{(R_{x+n} - R_{x+t}) - 5(M_{x+n+5}$$
$$+ M_{x+n+10} + \cdots + M_{x+t})\}] - \pi a_{x+n:\overline{t-n|}}$$

When $b' = b$, we have

$$\frac{1}{D_{x+n}}[(1+nb)(M_{x+n} - M_{x+t} + D_{x+t}) + b\{R_{x+n} - R_{x+t} + (t-n)(D_{x+t} - M_{x+t})\}]$$
$$- \pi a_{x+n:\overline{t-n|}}$$

π in these two formulas will have different values as found for the two benefits on page 303.

38. To find the value of an assurance with a compound reversionary bonus on similar conditions.

Whole-Life Policy.

$$\frac{1}{D_{x+n}}(1+5b)^{\frac{n}{5}}\{(M_{x+n} - M_{x+n+5}) + (1+5b)(M_{x+n+5} - M_{x+n+10}) + \cdots$$
$$+ b'(R_{x+n} - R_{x+n+5} - 5M_{x+n+5}) + b'(1+5b)(R_{x+n+5} - R_{x+n+10} - 5M_{x+n+10})$$
$$+ \cdots\} - \pi a_{x+n}$$

If $b' = b$,

$$\frac{1}{D_{x+n}}(1+5b)^{\frac{n}{5}}\{M_{x+n} + b(R_{x+n} - R_{x+n+5}) + b(1+5b)(R_{x+n+5} - R_{x+n+10})$$
$$+ b(1+5b)^2(R_{x+n+10} - R_{x+n+15}) + \cdots\} - \pi a_{x+n}$$

π will have a value appropriate to the benefit as indicated on page 304.

Endowment Assurance.

$$\frac{1}{D_{x+n}}(1+5b)^{\frac{n}{5}}\{(M_{x+n}-M_{x+n+5})+(1+5b)(M_{x+n+5}-M_{x+n+10})+\cdots$$

$$+(1+5b)^{\frac{t-n-5}{5}}(M_{x+t-5}-M_{x+t})+(1+5b)^{\frac{t-n}{5}}D_{x+t}$$

$$+b'(R_{x+n}-R_{x+n+5}-5M_{x+n+5})+b'(1+5b)(R_{x+n+5}-R_{x+n+10}-5M_{x+n+10})$$

$$+\cdots+b'(1+5b)^{\frac{t-n-5}{5}}(R_{x+t-5}-R_{x+t}-5M_{x+t})\}-\pi a_{x+n:\overline{t-n|}}$$

If $b'=b$,

$$\frac{1}{D_{x+n}}(1+5b)^{\frac{n}{5}}\{M_{x+n}+b(R_{x+n}-R_{x+n+5})+b(1+5b)(R_{x+n+5}-R_{x+n+10})+\cdots$$

$$+b(1+5b)^{\frac{t-n-5}{5}}(R_{x+t-5}-R_{x+t})+(1+5b)^{\frac{t-n}{5}}(D_{x+t}-M_{x+t})\}-\pi a_{x+n:\overline{t-n|}}$$

The values of π are indicated on page 305.

39. To find the reserve under a DISCOUNTED-BONUS or MINIMUM-PREMIUM policy.

(*a*) **Cash Bonus.**

It was found on page 306 that the deduction from the ordinary premium for a discounted bonus of k per annum of the premium was $kP'_x\frac{N_{x+2}}{N_{x-1}}$. Therefore in finding the reserve of this class of policy, we must add to the ordinary reserve for a full profit policy, to allow for this decrease in future premiums; but we must also make a deduction from the liability, in respect of future bonuses at this rate which will not be payable. That is

$$\begin{aligned}{}_nV'_x &= {}_nV_x+kP'_x\frac{N_{x+2}}{N_{x-1}}a_{x+n}-5kP'_x\frac{D_{x+n+5}+D_{x+n+10}+\cdots}{D_{x+n}}\\ &= {}_nV_x+kP'_x\frac{N_{x+2}}{N_{x-1}}a_{x+n}-kP'_x\frac{N_{x+n+2}}{N_{x+n-1}}a_{x+n}\\ &= {}_nV_x+kP'_x\left(\frac{N_{x+2}}{N_{x-1}}-\frac{N_{x+n+2}}{N_{x+n-1}}\right)a_{x+n}\end{aligned}$$

${}_nV_x$ being the reserve for a full profit policy.

(*b*) **Uniform Reversionary Bonus.**

The deduction from the ordinary premium was found to be $b\frac{R_x}{N_{x-1}}$. An addition and deduction must be made as above explained, and we shall have

$$
\begin{aligned}
{}_nV'_x &= {}_nV_x + b\frac{R_x}{N_{x-1}}a_{x+n} - b\frac{R_{x+n}}{D_{x+n}} \\
&= {}_nV_x - b\left(\frac{R_{x+n}}{N_{x+n-1}} - \frac{R_x}{N_{x-1}}\right)a_{x+n}
\end{aligned}
$$

(*c*) **Compound Reversionary Bonus.**

The deduction from the ordinary premium is

$$\frac{b(R_x - R_{x+5}) + b(1+5b)(R_{x+5} - R_{x+10}) + b(1+5b)^2(R_{x+10} - R_{x+15}) + \cdots}{N_{x-1}}.$$

The reserve accordingly will be

$$
\begin{aligned}
{}_nV'_x &= {}_nV_x + \frac{b(R_x - R_{x+5}) + b(1+5b)(R_{x+5} - R_{x+10}) + \cdots}{N_{x-1}}a_{x+n} \\
&\quad - \frac{b(R_{x+n} - R_{x+n+5}) + b(1+5b)(R_{x+n+5} - R_{x+n+10}) + \cdots}{D_{x+n}} \\
&= {}_nV_x - \left\{\frac{b(R_{x+n} - R_{x+n+5}) + b(1+5b)(R_{x+n+5} - R_{x+n+10}) + \cdots}{N_{x+n-1}}\right. \\
&\quad \left. - \frac{b(R_x - R_{x+5}) + b(1+5b)(R_{x+5} - R_{x+10}) + \cdots}{N_{x-1}}\right\}a_{x+n}
\end{aligned}
$$

40. The subject of surrender-values does not fall to be discussed, but it may be remarked that a surrender-value is as a rule granted only where a benefit is certainly payable, as under whole-life assurances, endowment assurances, joint-life assurances, etc. It is customary to allow no surrender-value where the benefit is only contingent, *e.g.*, temporary insurances, where the sum assured is payable only should the life die within the term, a contingency which may or may not happen; contingent insurance, (x) against (y), where the sum assured is payable only should (x) die before (y), which may or may not happen.

In the case of a pure endowment with return of the premiums in the event of previous death, a surrender-value is given, since a benefit is paid whatever happens; but where it is a pure endowment without return, usually no surrender-value is paid, as the benefit is then contingent on survivance.

In the case of the double-endowment assurance, the method by which the surrender-value is calculated requires special consideration, since it must not be overlooked that only half the benefit is certainly payable, the other half being contingent on survivance.

41. The principles of *Text Book*, Articles 122 and 123, on which formulas (56) and (57) are founded, may be stated generally so as to apply to any kind of benefit.

First, let W be the amount of paid-up policy to be granted. Then the value of a benefit of W must equal the value of the policy.

Second, the paid-up policy must equal the sum originally secured less that proportion of it which the future premiums will cover.

In the case of a whole-life assurance, we have

$$(1) \qquad W\,A_{x+n} = {}_nV_x$$

$$(2) \qquad W = 1 - \frac{P_x}{P_{x+n}}$$

Similarly for a deferred annuity-due, where $P = \dfrac{N_{x+t-1}}{N_{x-1} - N_{x+t-1}}$ we have

$$(1) \qquad W\frac{N_{x+t-1}}{D_{x+n}} = \frac{N_{x+t-1} - P(N_{x+n-1} - N_{x+t-1})}{D_{x+n}}$$

$$(2) \qquad W = 1 - \frac{P}{\dfrac{N_{x+t-1}}{N_{x+n-1} - N_{x+t-1}}}$$

For a reversionary annuity, where $P = \dfrac{a_x - a_{xy}}{1 + a_{xy}}$ we have

$$(1) \qquad W(a_{x+n} - a_{x+n:y+n}) = (a_{x+n} - a_{x+n:y+n}) - P(1 + a_{x+n:y+n})$$

$$(2) \qquad W = 1 - \frac{P}{\dfrac{a_{x+n} - a_{x+n:y+n}}{1 + a_{x+n:y+n}}}$$

42. To find the paid-up policy to be issued after n years in lieu of a pure endowment policy payable at the end of t years with return of premiums in event of previous death.

The value of the present contract is

$$\frac{\pi(N_{x-1}-N_{x+n-1})-\pi'(R_x-R_{x+n}-nM_{x+n})}{D_{x+n}}$$

Now, the n premiums paid are to be returned in event of death before age $x+t$ under the paid-up policy as under the original contract, and therefore we shall write

$$W\frac{D_{x+t}}{D_{x+n}}+\frac{n\pi'(M_{x+n}-M_{x+t})}{D_{x+n}} = \frac{\pi(N_{x-1}-N_{x+n-1})-\pi'(R_x-R_{x+n}-nM_{x+n})}{D_{x+n}}$$

whence $$W = \frac{\pi(N_{x-1}-N_{x+n-1})-\pi'(R_x-R_{x+n}-nM_{x+t})}{D_{x+t}}$$

43. Under a last-survivor assurance three cases arise in finding the paid-up policy.

(1) (x) and (y) both alive.

$$W\,A_{\overline{x+n:y+n}} = A_{\overline{x+n:y+n}}-P_{\overline{xy}}\,a_{\overline{x+n:y+n}}$$

$$\text{or}\quad W = 1-\frac{P_{\overline{xy}}}{P_{\overline{x+n:y+n}}}$$

(2) (x) dead.

$$W\,A_{y+n} = A_{y+n}-P_{\overline{xy}}\,a_{y+n}$$

$$\text{or}\quad W = 1-\frac{P_{\overline{xy}}}{P_{y+n}}$$

(3) (y) dead.

$$W\,A_{x+n} = A_{x+n}-P_{\overline{xy}}\,a_{x+n}$$

$$\text{or}\quad W = 1-\frac{P_{\overline{xy}}}{P_{x+n}}$$

44. It is a common practice for offices to guarantee paid-up policies under limited-payment whole-life assurances and endowment assurances, the amount of each paid-up policy bearing the same proportion to the original sum assured as the number of premiums paid bears to the whole number payable.

It sometimes happens however that the value of the guaranteed benefit exceeds the value of the original policy, and the conditions may be investigated.

Under the whole-life assurance by limited payments, where t premiums were originally payable and n have been paid, the amount of paid-up policy is $\frac{n}{t}$. Assuming the life to be still select, we have

$$\frac{n}{t}A_{[x+n]} > = < A_{[x+n]} - {}_tP_{[x]}a_{[x+n]:\overline{t-n}|}$$

according as

$${}_tP_{[x]}a_{[x+n]:\overline{t-n}|} > = < \left(1 - \frac{n}{t}\right)A_{[x+n]}$$

or according as

$${}_tP_{[x]} > = < \left(1 - \frac{n}{t}\right){}_{t-n}P_{[x+n]}$$

The following figures based on the $O^{[M]}$ Table at 3 per cent. illustrate the point:—

Age at Entry. (x)	Original Number of Payments (t)	${}_tP_{[x]}$	$\left(1 - \frac{n}{t}\right){}_{t-n}P_{[x+n]}$ when n = 5	10	15
30	20	2·639	2·670	2·698	2·702
40	15	3·953	3·975	3·965	...
50	15	4·963	4·920	4·781	...
60	10	8·271	7·859	...	...

Similarly, in the case of the endowment assurance

$$\frac{n}{t}A_{[x+n]:\overline{t-n}|} > = < A_{[x+n]:\overline{t-n}|} - P_{[x]\overline{t}|}a_{[x+n]:\overline{t-n}|}$$

according as

$$P_{[x]\overline{t}|} > = < \left(1 - \frac{n}{t}\right)P_{[x+n]:\overline{t-n}|}$$

In the endowment assurance of practice the right-hand side is always the greater, and therefore the value of the guaranteed

benefit is always less than the value of the original contract. The following figures are on the same basis as before.

Age at Entry. (x)	Endowment Term. (t)	$P_{[x]\overline{t}\vert}$	$\left(1-\frac{n}{t}\right)P_{[x+n]:\overline{t-n}\vert}$ when $n=$ 5	10	15
30	20	4·031	4·223	4·428	4·651
40	15	5·725	5·973	6·234	...
50	15	6·107	6·253	6·367	...
60	10	9·783	9·743	...	...

45. To convert a whole-life policy, effected by annual premiums n years ago at age x, into an endowment assurance payable at age $(x+n+t)$, or previous death.

Let the future yearly premium be P. Now the present value of the benefit as altered less the value of the future premiums must be equal to the reserve held by the office. That is,

$$A_{x+n:\overline{t}|} - P\,a_{x+n:\overline{t}|} = {}_nV_x$$

$$A_{x+n:\overline{t}|} - P_{x:\overline{n+t}|}\,a_{x+n:\overline{t}|} - P\,a_{x+n:\overline{t}|} + P_{x:\overline{n+t}|}\,a_{x+n:\overline{t}|} = {}_nV_x$$

$${}_nV_{x:\overline{n+t}|} - P\,a_{x+n:\overline{t}|} + P_{x:\overline{n+t}|}\,a_{x+n:\overline{t}|} = {}_nV_x$$

and $$P = \frac{{}_nV_{x:\overline{n+t}|} - {}_nV_x}{a_{x+n:\overline{t}|}} + P_{x:\overline{n+t}|}$$

Thus the future premium required is equal to the endowment assurance premium at the original age, together with the difference between the reserve required for the same endowment assurance if effected at the outset and the actual reserve in hand spread over the future duration of the policy.

Now $${}_nV_{x:\overline{n+t}|} = \frac{P_{x:\overline{n+t}|}(N_{x-1}-N_{x+n-1})-(M_x-M_{x+n})}{D_{x+n}}$$

and $${}_nV_x = \frac{P_x(N_{x-1}-N_{x+n-1})-(M_x-M_{x+n})}{D_{x+n}}$$

Therefore $${}_nV_{x:\overline{n+t}|} - {}_nV_x = \frac{(P_{x:\overline{n+t}|}-P_x)(N_{x-1}-N_{x+n-1})}{D_{x+n}}$$

and this is the lump sum to be paid to the office if the future premium is to be $P_{x:\overline{n+t}|}$, the premium for an endowment assurance at the original age at entry.

It is frequently suggested that it would be an equitable arrangement merely to pay the difference between the premiums accumulated at compound interest, or

$$(P_{x:\overline{n+t}|} - P_x)(1+i)s_{\overline{n}|}$$

But this is insufficient, since $\dfrac{N_{x-1} - N_{x+n-1}}{D_{x+n}} > (1+i)s_{\overline{n}|}$, as shown on page 295.

The following case will illustrate the absurdity of the suggestion. Suppose an office

(1) Permits its whole-life policy-holders to alter to endowment assurance on paying up merely the difference in premiums accumulated at interest, and

(2) Permits its endowment assurance policy-holders to pay only the whole-life premiums, the difference between the premiums being allowed to accumulate as a debt against the policy, which will be deducted from the sum assured on payment either at maturity or at previous death.

No policies will be taken out under the second scheme, for under the former just as good a benefit is secured for the same premium, with the additional advantage that the accumulated difference in the premiums will be deducted only at maturity and not at previous death. To put the two classes on an equality, the amount to be charged at the end of m years in the first scheme should be altered to

$$(P_{x\overline{m}|} - P_x)\ \frac{N_{x-1} - N_{x+m-1}}{D_{x+m}}$$

Another illustration of the same absurdity may be given. Suppose the assumed conditions to apply to all classes of assurance. The life assured would then be well advised to take out a temporary assurance which will give him the sum assured in the event of death during the term at the lowest premium. Then at the end of the term, if he survives, he alters to endowment assurance, and pays the accumulated difference

$$(P_{x\overline{m}|} - P^1_{x\overline{m}|})(1+i)s_{\overline{m}|} = P_{x\overline{m}|}^{\ \ 1}(1+i)s_{\overline{m}|}$$

and he should immediately receive 1, which is clearly wrong.

The correct amount to be paid by him is

$$(P_{\overline{xm}|} - P^{1}_{\overline{xm}|})\frac{N_{x-1} - N_{x+m-1}}{D_{x+m}} = P^{1}_{\overline{xm}|}\frac{N_{x-1} - N_{x+m-1}}{D_{x+m}}$$
$$= 1$$

Thus he provides the payment of 1, which has to be made to him.

From the formula given on page 387, we have

$$P = P_{x:\overline{n+t}|} + \frac{{}_nV_{x:\overline{n+t}|} - {}_nV_x}{a_{x+n:\overline{t}|}}$$

$$= P_{x:\overline{n+t}|} + \frac{(P_{x+n:\overline{t}|} - P_{x:\overline{n+t}|})a_{x+n:\overline{t}|} - {}_nV_x}{a_{x+n:\overline{t}|}}$$

$$= P_{x+n:\overline{t}|} - \frac{{}_nV_x}{a_{x+n:\overline{t}|}}$$

From this expression we observe that the future premium to be paid is the endowment assurance premium at the present age, less the reserve value of the existing policy spread over the future premiums.

Again, we may argue as follows:—As no premiums are to be received after age $(x+n+t)$, and as the office will lose the interest in advance after that age on the assurance of 1 which will then be payable, the total loss to the office is

$$(P_x + d)\frac{N_{x+n+t-1}}{D_{x+n}}$$

$$= (P_x + d)(a_{x+n} - a_{x+n:\overline{t}|})$$

$$= \frac{a'_{x+n} - a_{x+n:\overline{t}|}}{a_x}$$

Spreading this loss over the future premiums and adding the result to the present premium, we have

$$P = P_x + \frac{a_{x+n} - a_{x+n:\overline{t}|}}{a_x \times a_{x+n:\overline{t}|}}$$

Now let the premium to be paid remain at P_x; to find the altered sum assured, S.

We have

$$SA_{x+n:\overline{t|}} - P_x a_{x+n:\overline{t|}} = {}_nV_x$$

$$S = \frac{{}_nV_x + P_x a_{x+n:\overline{t|}}}{A_{x+n:\overline{t|}}}$$

$$= \frac{{}_nV_x}{A_{x+n:\overline{t|}}} + \frac{P_x}{P_{x+n:\overline{t|}}}$$

which may be explained by general reasoning. $A_{x+n:\overline{t|}}$ is the single premium at the present age for a sum assured of 1 payable at age $x+n+t$ or previous death. Therefore ${}_nV_x$ is the single premium for a sum assured of $\frac{{}_nV_x}{A_{x+n:\overline{t|}}}$. Also $P_{x+n:\overline{t|}}$ is the annual premium for a similar assurance of 1, and therefore P_x is the annual premium for a sum assured of $\frac{P_x}{P_{x+n:\overline{t|}}}$. Together these two sums make the total sum assured under the policy as altered.

46. It is sometimes desired to apply the bonus on a whole-life policy so as to limit the future premiums or to alter the policy into an endowment assurance.

Mr Manly has put forward the following formulas in answer to this problem, and has also supplied tables to facilitate their application in practice.

If the future premiums are to be limited, let x_1, x_2, x_3, etc., be the ages of the life assured at the successive periods of division of profits; y_1, y_2, y_3, etc., the ages, at and after which the premiums are to cease; and B_1, B_2, B_3, etc., the amount of the reversionary bonuses.

Then at the first investigation, we have

$$B_1 A_{x_1} = \frac{P_x N_{y_1-1}}{D_{x_1}}$$

from which y_1 may be obtained.

At the succeeding investigation, we have

$$B_2A_{x_2} = \frac{P_x(N_{y_2-1} - N_{y_1-1})}{D_{x_2}}$$

from which y_2 may be obtained.

And we may proceed similarly at the following investigations.

If the policy is to be altered into an endowment assurance, let y_1, y_2, y_3, etc., be the ages at which the sum assured will be payable, and let the other symbols have the same meaning as before.

We have at the first investigation

$$B_1A_{x_1} = (P_x + d)\frac{N_{y_1-1}}{D_{x_1}}$$

from which we may obtain y_1.

At the next investigation

$$B_2A_{x_2} = (P_x + d)\frac{N_{y_2-1} - N_{y_1-1}}{D_{x_2}}$$

from which y_2 may be found.

And we may proceed similarly at the succeeding investigations.

EXAMPLES

1. Given $P_{25} = \cdot01521$, $P_{42} = \cdot02654$, $a_{42} = 15\cdot5679$, find the value of a whole-life policy for £1500 effected at age 25, which has been 17 years in force, and to which a reversionary bonus of £383 is attached.

$$\begin{aligned} \text{Value of policy} &= 1883 \times A_{42} - 1500 \times P_{25}(1 + a_{42}) \\ &= (1883 \times P_{42} - 1500 \times P_{25})(1 + a_{42}) \\ &= (1883 \times \cdot02654 - 1500 \times \cdot01521)(1 + 15\cdot5679) \\ &= (49\cdot975 - 22\cdot815)16\cdot5679 \\ &= 449\cdot984, \text{ say £449, 19s. 8d.} \end{aligned}$$

2. Give as many formulas as you know for ${}_nV_{\overline{xt}|}$

$$
\begin{aligned}
{}_nV_{\overline{xt}|} &= A_{x+n:\overline{t-n}|} - P_{\overline{xt}|}(1 + a_{x+n:\overline{t-n-1}|}) \\
&= \frac{P_{\overline{xt}|}(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}} \\
&= 1 - (P_{\overline{xt}|} + d)(1 + a_{x+n:\overline{t-n-1}|}) \\
&= (P_{x+n:\overline{t-n}|} - P_{\overline{xt}|})(1 + a_{x+n:\overline{t-n-1}|}) \\
&= A_{x+n:\overline{t-n}|}\left(1 - \frac{P_{\overline{xt}|}}{P_{x+n:\overline{t-n}|}}\right) \\
&= 1 - \frac{1 + a_{x+n:\overline{t-n-1}|}}{1 + a_{x:\overline{t-1}|}} \\
&= \frac{a_{x:\overline{t-1}|} - a_{x+n:\overline{t-n-1}|}}{1 + a_{x:\overline{t-1}|}} \\
&= 1 - \frac{1 - A_{x+n:\overline{t-n}|}}{1 - A_{\overline{xt}|}} \\
&= \frac{A_{x+n:\overline{t-n}|} - A_{\overline{xt}|}}{1 - A_{\overline{xt}|}} \\
&= 1 - \frac{P_{\overline{xt}|} + d}{P_{x+n:\overline{t-n}|} + d} \\
&= \frac{P_{x+n:\overline{t-n}|} - P_{\overline{xt}|}}{P_{x+n:\overline{t-n}|} + d} \\
&= 1 - (1 - {}_1V_{\overline{xt}|})(1 - {}_1V_{x+1:\overline{t-1}|})\cdots(1 - {}_1V_{x+n-1:\overline{t-n+1}|})
\end{aligned}
$$

3. Find formulas for ${}_nV_x$, ${}_nV_{\overline{xt}|}$, and ${}_{n:t}V_x$, when the rate of interest is zero.

$$
\begin{aligned}
{}_nV_x &= 1 - \frac{1 + a_{x+n}}{1 + a_x} \\
&= \text{(when interest is zero) } 1 - \frac{1 + e_{x+n}}{1 + e_x}
\end{aligned}
$$

$$_{n}V_{\overline{x t|}} = 1 - \frac{1 + a_{x+n:\overline{t-n-1|}}}{1 + a_{x:\overline{t-1|}}}$$

$$= \text{(when interest is zero) } 1 - \frac{1 + e_{x+n:\overline{t-n-1|}}}{1 + e_{x:\overline{t-1|}}}$$

$$_{n:t}V_x = A_{x+n} - {}_tP_x(1 + a_{x+n:\overline{t-n-1|}})$$

$$= \text{(when interest is zero) } 1 - \frac{1 + e_{x+n:\overline{t-n-1|}}}{1 + e_{x:\overline{t-1|}}}$$

since $\quad {}_tP_x = \dfrac{A_x}{1 + a_{x:\overline{t-1|}}} = \dfrac{1}{1 + e_{x:\overline{t-1|}}}$ when interest is zero.

It will be observed that the formula for limited-payment whole-life assurance is the same as for endowment assurance, so long as $n < t$ in the limited-payment assurance.

4. Under a policy taken out at age x which has been n years in force, the sum assured and bonuses amount to S. Prove that the value of the policy is equal to $\left(S + P_x + \dfrac{P_x}{i}\right)A_{x+n} - \left(P_x + \dfrac{P_x}{i}\right)$

$$\text{Value of policy} = SA_{x+n} - P_x(1 + a_{x+n})$$

$$= SA_{x+n} - \frac{P_x(1 - A_{x+n})}{d}$$

$$= SA_{x+n} - P_x(1 - A_{x+n})\left(1 + \frac{1}{i}\right)$$

$$= \left(S + P_x + \frac{P_x}{i}\right)A_{x+n} - \left(P_x + \frac{P_x}{i}\right)$$

5. There are 44 policies of 1 each, all effected at age 40, which have been in force 1, 2, 3, etc., up to 44 years respectively. The sum of their values is 17·52789. Find the values separately of the aggregate sums assured, and of the future net premiums, having given $A_{40} = \cdot 379434$.

By *Text Book*, formula 18, we have

$$\Sigma A_{x+n} = \Sigma V_x(1 - A_x) + rA_x$$

Therefore

$$\begin{aligned}\Sigma A_{40+n} &= \Sigma V_{40}(1-A_{40})+44A_{40}\\ &= (17\cdot52789\times\cdot620566)+(44\times\cdot379434)\\ &= 10\cdot87721+16\cdot69510\\ &= 27\cdot57231\end{aligned}$$

which is the value of the aggregate sums assured. And the value of the future net premiums

$$\begin{aligned}&= \Sigma A_{40+n}-\Sigma V_{40}\\ &= 27\cdot57231-17\cdot52789\\ &= 10\cdot04442\end{aligned}$$

6. Explain under what circumstances ${}_nV_x < {}_{n-1}V_{x+1}$, and give an example from some known table of mortality where the anomaly occurs.

$${}_nV_x < {}_{n-1}V_{x+1}$$

$$\text{if} \qquad 1-\frac{\mathbf{a}_{x+n}}{\mathbf{a}_x} < 1-\frac{\mathbf{a}_{x+n}}{\mathbf{a}_{x+1}}$$

$$\text{if} \qquad \frac{\mathbf{a}_x}{\mathbf{a}_{x+n}} < \frac{\mathbf{a}_{x+1}}{\mathbf{a}_{x+n}}$$

$$\text{if} \qquad a_x < a_{x+1}$$

which is very unusual, but does occur at the infantile ages of all mortality tables, and at very advanced ages of some mortality tables which have been badly graduated, *e.g.* the Carlisle Table.

7. Find the reserve value after n years of a pure endowment policy effected at age x to be payable at age $(x+t)$, with premiums limited to r and to be returned in event of death before age $(x+t)$.

The net premium for such a policy was found on page 324.

The reserve value after n years is

(1) $n < r$.

Prospectively

$$\frac{D_{x+t}}{D_{x+n}}+\{\pi(1+\kappa)+c\}\frac{nM_{x+n}+R_{x+n}-R_{x+r}-rM_{x+t}}{D_{x+n}}$$
$$-\pi\frac{N_{x+n-1}-N_{x+r-1}}{D_{x+n}}$$

Retrospectively

$$\pi\,\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}-\{\pi(1+\kappa)+c\}\,\frac{R_x-R_{x+n}-nM_{x+n}}{D_{x+n}}$$

(2) $n =$ or $> r$.

Prospectively

$$\frac{D_{x+t}}{D_{x+n}}+r\{\pi(1+\kappa)+c\}\,\frac{M_{x+n}-M_{x+t}}{D_{x+n}}$$

Retrospectively

$$\pi\,\frac{N_{x-1}-N_{x+r-1}}{D_{x+n}}-\{\pi(1+\kappa)+c\}\,\frac{R_x-R_{x+r}-rM_{x+n}}{D_{x+n}}$$

In each case the values found by the two methods may be proved equal, and it will form a useful exercise to do this as shown for other similar problems.

8. Find the annual premium to secure an endowment assurance to (x), maturing at the end of t years, with a guaranteed bonus of £2 per cent. for each year completed; and also the reserve value at the end of n years.

Benefit side

$$\frac{M_x-M_{x+t}+D_{x+t}}{D_x}+\cdot02\,\frac{R_{x+1}-R_{x+t}-(t-1)M_{x+t}+tD_{x+t}}{D_x}$$

Payment side

$$P\,\frac{N_{x-1}-N_{x+t-1}}{D_x}$$

whence equating and solving

$$P=\frac{M_x-M_{x+t}+D_{x+t}+\cdot02\{R_{x+1}-R_{x+t}-(t-1)M_{x+t}+tD_{x+t}\}}{N_{x-1}-N_{x+t-1}}$$

The reserve value after n years is

Prospectively

$$\frac{M_{x+n}-M_{x+t}+D_{x+t}}{D_{x+n}}+\cdot02\,\frac{nM_{x+n}+R_{x+n+1}-R_{x+t}-(t-1)M_{x+t}+tD_{x+t}}{D_{x+n}}$$

$$-P\,\frac{N_{x+n-1}-N_{x+t-1}}{D_{x+n}}$$

Retrospectively

$$P\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}-\frac{M_x-M_{x+n}}{D_{x+n}}-\cdot 02\,\frac{R_{x+1}-R_{x+n}-(n-1)M_{x+n}}{D_{x+n}}$$

9. Find in terms of commutation columns the value after n years of a policy for a deferred annuity, payable half-yearly, maintained by annual premiums, to be entered on at age $(x+z)$. Examine the three cases, $n <, =, > z$.

Here $\quad \pi = \dfrac{{}_{z|}a_x^{(2)}}{1+a_{x:\overline{z-1}|}} = \dfrac{N_{x+z}+\frac{1}{4}D_{x+z}}{N_{x-1}-N_{x+z-1}}$

and reserve value of policy

(1) $n < z$.

(a) Prospectively $\quad \dfrac{N_{x+z}+\frac{1}{4}D_{x+z}}{D_{x+n}} - \pi\dfrac{N_{x+n-1}-N_{x+z-1}}{D_{x+n}}$

(b) Retrospectively $\quad \pi\dfrac{N_{x-1}-N_{x+n-1}}{D_{x+n}}$

(2) $n = z$.

(a) Prospectively $\quad \dfrac{N_{x+z}+\frac{1}{4}D_{x+z}}{D_{x+z}}$

(b) Retrospectively $\quad \pi\dfrac{N_{x-1}-N_{x+z-1}}{D_{x+z}}$

(3) $n > z$.

(a) Prospectively $\quad \dfrac{N_{x+n}+\frac{1}{4}D_{x+n}}{D_{x+n}}$

(b) Retrospectively $\quad \pi\dfrac{N_{x-1}-N_{x+z-1}}{D_{x+n}} - \dfrac{(N_{x+z}-N_{x+n})+\frac{1}{4}(D_{x+z}-D_{x+n})}{D_{x+n}}$

10. What annual premium should be charged for an assurance of £1000 to (x), the premium being successively reduced by $\frac{1}{10}$ of the first premium and ceasing altogether after the tenth payment? What is the reserve value of the policy at the end of 6 years?

Benefit side $= \dfrac{1000\,M_x}{D_x}$

Payment side $= P\,\dfrac{N_{x-1} - \frac{1}{10}(S_x - S_{x+10})}{D_x}$

whence $P = \dfrac{1000M_x}{N_{x-1} - \frac{1}{10}(S_x - S_{x+10})}$

The reserve value at the end of 6 years

$$= 1000A_{x+6} - P\,\frac{\frac{4}{10}N_{x+5} - \frac{1}{10}(S_{x+6} - S_{x+10})}{D_{x+6}}$$

11. Find the single and annual premiums for an assurance upon a life aged x, the sum assured increasing in amount at compound interest at rate j. What would be the policy-value after t years?

The single premium is

$$\frac{(1+j)M_x + j(1+j)M_{x+1} + j(1+j)^2M_{x+2} + \cdots}{D_x}$$

And the value of this policy after t years is

$$(1+j)^t\,\frac{(1+j)M_{x+t} + j(1+j)M_{x+t+1} + j(1+j)^2M_{x+t+2} + \cdots}{D_{x+t}}$$

To find the annual premium, we have the benefit side the same as the single premium above, and

Payment side $= P\,\dfrac{N_{x-1}}{D_x}$

Hence $P = \dfrac{(1+j)M_x + j(1+j)M_{x+1} + j(1+j)^2M_{x+2} + \cdots}{N_{x-1}}$

The value of this policy after t years is

$$(1+j)^t\,\frac{(1+j)M_{x+t} + j(1+j)M_{x+t+1} + j(1+j)^2M_{x+t+2} + \cdots}{D_{x+t}}$$

$$-P\,\frac{N_{x+t-1}}{D_{x+t}}$$

Or again, the single premium is

$$\frac{v(1+j)d_x + v^2(1+j)^2 d_{x+1} + \cdots}{l_x}$$

Find J, such that $\frac{1}{1+J} = \frac{1+j}{1+i}$. Then the single premium $= A'_x$ calculated at rate J, and the value of the policy after t years is $(1+j)^t A'_{x+t}$.

In the case of the annual-premium policy we have $P = \frac{A'_x}{1+a_x}$. The value of this policy after t years is $(1+j)^t A'_{x+t} - P a_{x+t}$.

12. A policy by annual premium was issued n years ago for a reversionary annuity to (x) after (y). Find the reserve value at the present time.

(1) (y) still alive $\quad {}_nVa_{y|x} = a_{y+n|x+n} - Pa_{y|x}(1 + a_{x+n:y+n})$

(2) (y) dead $\quad {}_nVa_{y|x} = a_{x+n}$

13. Find an expression for the value at the end of n years of a contingent insurance payable if the survivor of (x) and (y) die before (z).

It must be ascertained whether (x) and (y) are both alive. Then if both are alive

$${}_nV_{\overline{xy}:z}^{1} = A_{\overline{x+n:y+n}:z+n}^{\;\;1} - P_{\overline{xy}:z}^{1}(1 + a_{\overline{x+n:y+n}:z+n})$$

If (x) is dead,

$${}_nV_{\overline{xy}:z}^{1} = A_{y+n:z+n}^{1} - P_{\overline{xy}:z}^{1}(1 + a_{y+n:z+n})$$

And, similarly, if (y) is dead,

$${}_nV_{\overline{xy}:z}^{1} = A_{x+n:z+n}^{1} - P_{\overline{xy}:z}^{1}(1 + a_{x+n:z+n})$$

14. Obtain by the retrospective method a formula in terms of annual premiums for ${}_nV_{x\overline{t}|}^{1}$, and prove by general reasoning and algebraically that

$${}_nV_x = \frac{P_x - P_{x\overline{n}|}^{1}}{P_{x\overline{n}|}^{\;\;1}}$$

$$ {}_nV^1_{x\overline{t|}} = \frac{P^1_{x\overline{t|}}(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}} $$

$$ = \frac{\dfrac{P^1_{x\overline{t|}} - \dfrac{M_x - M_{x+n}}{N_{x-1} - N_{x+n-1}}}{D_{x+n}}}{N_{x-1} - N_{x+n-1}} $$

$$ = \frac{P^1_{x\overline{t|}} - P^1_{x\overline{n|}}}{P_{x\overline{n|}}^{\ \ 1}} $$

Similarly, $$ {}_nV_x = \frac{P_x(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}} $$

$$ = \frac{P_x - P^1_{x\overline{n|}}}{P_{x\overline{n|}}^{\ \ 1}} $$

These results are correct, for the risk already undergone is that of a temporary assurance for n years, which would be covered by an annual premium of $P^1_{x\overline{n|}}$; the surplus premiums paid are therefore $(P^1_{x\overline{t|}} - P^1_{x\overline{n|}})$ and $(P_x - P^1_{x\overline{n|}})$ respectively, which obviously must have sufficed to purchase certain amounts of endowment at age $(x+n)$. Now, $P_{x\overline{n|}}^{\ \ 1}$ will secure an endowment of 1; therefore $(P^1_{x\overline{t|}} - P^1_{x\overline{n|}})$ and $(P_x - P^1_{x\overline{n|}})$ will secure endowments of

$$ \frac{P^1_{x\overline{t|}} - P^1_{x\overline{n|}}}{P_{x\overline{n|}}^{\ \ 1}} \text{ and } \frac{P_x - P^1_{x\overline{n|}}}{P_{x\overline{n|}}^{\ \ 1}} \text{ respectively.} $$

These, then, must be the accumulated reserves under the two policies.

Likewise, in the case of the endowment assurance we should have

$$ {}_nV_{x\overline{t|}} = \frac{P_{x\overline{t|}} - P^1_{x\overline{n|}}}{P_{x\overline{n|}}^{\ \ 1}} $$

and when $n = t$,

$$ {}_tV_{x\overline{t|}} = \frac{P_{x\overline{t|}} - P^1_{x\overline{t|}}}{P_{x\overline{t|}}^{\ \ 1}} $$

$$ = 1 $$

showing that the accumulation of overpayments will exactly amount to the sum required at maturity.

15. Given at $3\frac{1}{2}$ per cent. ${}_nV_x = \cdot 50084$, and $P_{x+n} = \cdot 05917$, find the premium for the age at entry (P_x), the value of the sum assured (A_{x+n}), and the amount of paid-up policy equivalent to ${}_nV_x$.

Since
$$ {}_nV_x = \frac{P_{x+n} - P_x}{P_{x+n} + d} $$

therefore
$$ \begin{aligned} P_x &= P_{x+n}(1 - {}_nV_x) - d\,{}_nV_x \\ &= \cdot 05917 \times \cdot 49916 - \cdot 03148 \times \cdot 50084 \\ &= \cdot 01377 \end{aligned} $$

$$ \begin{aligned} A_{x+n} &= \frac{P_{x+n}}{P_{x+n} + d} \\ &= \frac{\cdot 05917}{\cdot 05917 + \cdot 03148} \\ &= \cdot 65273 \end{aligned} $$

$$ \begin{aligned} W &= \frac{{}_nV_x}{A_{x+n}} \\ &= \frac{\cdot 50084}{\cdot 65273} \\ &= \cdot 76730 \end{aligned} $$

16. Having given that at 4 per cent. interest, $P_{20:\overline{40}|} = \cdot 01615$, and $A_{40:\overline{20}|} = \cdot 51078$, find ${}_{20}V_{20:\overline{40}|}$ and ${}_{20}(FP)_{20:\overline{40}|}$

$$ \begin{aligned} {}_{20}V_{20:\overline{40}|} &= A_{40:\overline{20}|} - P_{20:\overline{40}|}\,a_{40:\overline{20}|} \\ &= \cdot 51078 - \cdot 01615 \times \frac{1 - \cdot 51078}{\cdot 03846} \quad \text{since } a = \frac{1 - A}{d} \\ &= \cdot 51078 - \cdot 20543 \\ &= \cdot 30535 \end{aligned} $$

$$ \begin{aligned} {}_{20}(FP)_{20:\overline{40}|} &= \frac{{}_{20}V_{20:\overline{40}|}}{A_{40:\overline{20}|}} \\ &= \frac{\cdot 30535}{\cdot 51078} \\ &= \cdot 59781 \end{aligned} $$

17. A whole-life policy was effected n years ago at age x, and the sum assured is now to be reduced by half, the value of the rest of the policy being applied in reduction of the annual premium. Find the future annual premium.

(*a*) We may take half the premium formerly payable and deduct therefrom the value of half the policy divided by the annuity-due for the rest of life.

$$\text{Thus,}\quad \tfrac{1}{2}P_x - \frac{\tfrac{1}{2}\,{}_nV_x}{a_{x+n}} = \tfrac{1}{2}P_x - \tfrac{1}{2}(P_{x+n} - P_x)$$

$$= P_x - \tfrac{1}{2}P_{x+n}$$

(*b*) Or we may take the premium at the present age for a policy of $\frac{1}{2}$, and deduct therefrom the whole value of the old policy divided by the annuity-due at the present age.

$$\text{Thus,}\quad \tfrac{1}{2}P_{x+n} - \frac{{}_nV_x}{a_{x+n}} = \tfrac{1}{2}P_{x+n} - (P_{x+n} - P_x)$$

$$= P_x - \tfrac{1}{2}P_{x+n}$$

(*c*) Or we may reason that the value of the present policy must equal the value of a sum assured of $\frac{1}{2}$ less the value of the future premiums. Thus, if P be the future annual premium,

$$ {}_nV_x = \tfrac{1}{2}A_{x+n} - Pa_{x+n}$$

$$\text{and}\quad Pa_{x+n} = \tfrac{1}{2}A_{x+n} - {}_nV_x$$

$$= \{\tfrac{1}{2}P_{x+n} - (P_{x+n} - P_x)\}a_{x+n}$$

$$\text{hence}\quad P = P_x - \tfrac{1}{2}P_{x+n}$$

18. (x) and (y), who are insured under a joint-life policy for £1000, desire at the end of n years to have it converted into two single-life assurances for £500 each. What premiums will be payable by (x) and (y) respectively?

The joint-life policy has acquired a value of ${}_nV_{xy}$ of which (x)'s share is ${}_nV_{\overset{1}{x}y}$ and (y)'s ${}_nV_{x\overset{1}{y}}$, these two parts together making up the whole value of the policy. Now, if from the premium payable by (x) at his present age for a £500 policy we deduct his interest in the old policy-value spread over the whole

of his life, we shall arrive at an equitable premium to be paid by him. Thus we have for (x)'s new policy for £500 a premium of

$$500P_{x+n} - \frac{1000\,{}_nV^{\,1}_{xy}}{a_{x+n}}$$

And similarly for (y)'s new policy the premium is

$$500P_{y+n} - \frac{1000\,{}_nV^{1}_{xy}}{a_{y+n}}$$

19. A whole-life policy for £1000, effected at age 20, has been thirty years in force, and has accumulated bonus additions of £450. It is proposed to devote part of the bonus to convert the policy (including the remainder of the declared bonus) into an endowment assurance maturing at age 65. Using the H^{M} table at 3 per cent., and ignoring the question of loading, find how much bonus will remain attached to the policy after the alteration has been made.

The office has in hand at present $1000\,{}_{30}V_{20} + 450A_{50}$. If X be the amount of bonus to be surrendered this will be reduced to $1000\,{}_{30}V_{20} + (450 - X)A_{50}$, and it will require to have in hand for the new contract

$$1000(A_{50:\overline{15}|} - P_{20}a_{50:\overline{15}|}) + (450 - X)A_{50:\overline{15}|}$$

The difference between these two reserves must be the present value of the bonus to be surrendered. That is

$$XA_{50} = 1000(A_{50:\overline{15}|} - P_{20}a_{50:\overline{15}|}) - 1000\,{}_{30}V_{20} + (450 - X)(A_{50:\overline{15}|} - A_{50})$$

$$X = \frac{1000A_{50:\overline{15}|} - 1000P_{20}a_{50:\overline{15}|} - 1000\,{}_{30}V_{20} + 450(A_{50:\overline{15}|} - A_{50})}{A_{50:\overline{15}|}}$$

$$= \frac{685{\cdot}47 - 154{\cdot}10 - 353{\cdot}53 + 53{\cdot}70}{{\cdot}68547}$$

$$= 337{\cdot}783$$

$$= \text{£}337,\ 15\text{s. } 8\text{d. approximately.}$$

Bonus amounting to £112, 4s. 4d. will therefore remain attached to the policy.

At future divisions of surplus the whole-life bonus only will be applicable to the policy, and it will have to be converted into endowment assurance bonus by simple proportion.

20. Give the prospective and retrospective values after n years of a child's endowment policy effected at age x, payable at age $(x+t)$, with premiums to be payable only so long as the father (y) lived along with (x) and to be returnable in event of death of (x) before age $(x+t)$. Prove the formulas identical.

Prospectively—

If the father is alive to-day, which will happen in $l_{x+n} \times {}_n p_y$ cases of the survivors,

$$\frac{D_{x+t}}{D_{x+n}} + \frac{\{\pi(1+\kappa)+c\}}{D_{x+n}}\Big\{(n+1)(M_{x+n} - M_{x+t}) + p_{y+n}(M_{x+n+1} - M_{x+t}) + \cdots$$
$$+ {}_{t-n-1}p_{y+n}(M_{x+t-1} - M_{x+t})\Big\} - \pi\frac{N_{x+n-1:y+n-1} - N_{x+t-1:y+t-1}}{D_{x+n:y+n}}$$

If the father died after n payments: $l_{x+n}({}_{n-1}p_y - {}_n p_y)$ cases,

$$\frac{D_{x+t}}{D_{x+n}} + \{\pi(1+\kappa)+c\}\frac{n(M_{x+n} - M_{x+t})}{D_{x+n}}$$

If the father died after $(n-1)$ payments: $l_{x+n}({}_{n-2}p_y - {}_{n-1}p_y)$ cases,

$$\frac{D_{x+t}}{D_{x+n}} + \{\pi(1+\kappa)+c\}\frac{(n-1)(M_{x+n} - M_{x+t})}{D_{x+n}}$$

etc. etc.

If the father died after 1 payment: $l_{x+n}(1 - p_y)$ cases,

$$\frac{D_{x+t}}{D_{x+n}} + \{\pi(1+\kappa)+c\}\frac{M_{x+n} - M_{x+t}}{D_{x+n}}$$

The sum of all these cases is

$$l_{x+n}\Big[\frac{D_{x+t}}{D_{x+n}} + \{\pi(1+\kappa)+c\}\Big(\frac{M_{x+n} - M_{x+t}}{D_{x+n}} + p_y\frac{M_{x+n} - M_{x+t}}{D_{x+n}} + {}_2p_y\frac{M_{x+n} - M_{x+t}}{D_{x+n}}$$
$$+ \cdots + {}_n p_y\frac{M_{x+n} - M_{x+t}}{D_{x+n}} + {}_{n+1}p_y\frac{M_{x+n+1} - M_{x+t}}{D_{x+n}} + \cdots + {}_{t-1}p_y\frac{M_{x+t-1} - M_{x+t}}{D_{x+n}}\Big)$$
$$- \pi\,{}_n p_y\frac{N_{x+n-1:y+n-1} - N_{x+t-1:y+t-1}}{D_{x+n:y+n}}\Big]$$

Retrospectively—

$$\pi\{l_x(1+i)^n + l_{x+1}\times p_y(1+i)^{n-1} + l_{x+2}\times {}_2p_y(1+i)^{n-2} + \dots$$

$$+ l_{x+n-1}\times {}_{n-1}p_y(1+i)\} - \{\pi(1+\kappa)+c\}\Big(\frac{M_x - M_{x+n}}{v^{x+n}} + p_y\frac{M_{x+1} - M_{x+n}}{v^{x+n}}$$

$$+ {}_2p_y\frac{M_{x+2} - M_{x+n}}{v^{x+n}} + \dots + {}_{n-1}p_y\frac{M_{x+n-1} - M_{x+n}}{v^{x+n}}\Big)$$

$$= \pi\frac{N_{x-1:y-1} - N_{x+n-1:y+n-1}}{l_y v^{x+n}} - \{\pi(1+\kappa)+c\}\Big(\frac{M_x - M_{x+n}}{v^{x+n}} + p_y\frac{M_{x+1} - M_{x+n}}{v^{x+n}}$$

$$+ {}_2p_y\frac{M_{x+2} - M_{x+n}}{v^{x+n}} + \dots + {}_{n-1}p_y\frac{M_{x+n-1} - M_{x+n}}{v^{x+n}}\Big)$$

To prove the two final expressions identical, we must first recall the expression from which the value of the premium was obtained, page 315. Adapting that expression to the present case, we have

$$\frac{D_{x+t}}{D_x} + \{\pi(1+\kappa)+c\}\Big(\frac{M_x - M_{x+t}}{D_x} + p_y\frac{M_{x+1} - M_{x+t}}{D_x} + {}_2p_y\frac{M_{x+2} - M_{x+t}}{D_x} + \dots$$

$$+ {}_{t-1}p_y\frac{M_{x+t-1} - M_{x+t}}{D_x}\Big)$$

$$= \pi\frac{N_{x-1:y-1} - N_{x+t-1:y+t-1}}{D_{xy}}$$

Therefore

$$D_{x+t} + \{\pi(1+\kappa)+c\}\{(M_x - M_{x+t}) + p_y(M_{x+1} - M_{x+t}) + {}_2p_y(M_{x+2} - M_{x+t}) + \dots$$

$$+ {}_{t-1}p_y(M_{x+t-1} - M_{x+t})\}$$

$$= \pi(N_{x-1:y-1} - N_{x+t-1:y+t-1})\frac{D_x}{D_{xy}}$$

Also it is to be noted that D_{xy} is of the form $v^x l_x l_y$.

Now the reserve by the prospective method

$$= l_{x+n}\Big[\frac{D_{x+t}}{D_{x+n}} + \{\pi(1+\kappa)+c\}\Big(\frac{M_{x+n}-M_{x+t}}{D_{x+n}} + p_y\frac{M_{x+n}-M_{x+t}}{D_{x+n}}$$

$$+ {}_2p_y\frac{M_{x+n}-M_{x+t}}{D_{x+n}} + \cdots + {}_np_y\frac{M_{x+n}-M_{x+t}}{D_{x+n}} + {}_{n+1}p_y\frac{M_{x+n+1}-M_{x+t}}{D_{x+n}} + \cdots$$

$$+ {}_{t-1}p_y\frac{M_{x+t-1}-M_{x+t}}{D_{x+n}}\Big) - \pi\,{}_np_y\frac{N_{x+n-1:y+n-1}-N_{x+t-1:y+t-1}}{D_{x+n:y+n}}\Big]$$

$$= \frac{1}{v^{x+n}}\Big[D_{x+t} + \{\pi(1+\kappa)+c\}\{(M_{x+n}-M_{x+t}) + p_y(M_{x+n}-M_{x+t}) + \cdots$$

$$+ {}_np_y(M_{x+n}-M_{x+t}) + {}_{n+1}p_y(M_{x+n+1}-M_{x+t}) + \cdots$$

$$+ {}_{t-1}p_y(M_{x+t-1}-M_{x+t})\} - \pi(N_{x+n-1:y+n-1}-N_{x+t-1:y+t-1})\frac{D_{x+n}}{D_{x+n:y+n}}\frac{l_{y+n}}{l_y}\Big]$$

$$= \frac{1}{v^{x+n}}\Big[D_{x+t} + \{\pi(1+\kappa)+c\}\{(M_x-M_{x+t}) + p_y(M_{x+1}-M_{x+t})$$

$$+ {}_2p_y(M_{x+2}-M_{x+t}) + \cdots + {}_{t-1}p_y(M_{x+t-1}-M_{x+t})\}$$

$$-\pi(N_{x-1:y-1}-N_{x+t-1:y+t-1})\frac{D_x}{D_{xy}} + \pi(N_{x-1:y-1}-N_{x+n-1:y+n-1})\frac{1}{l_y}$$

$$-\{\pi(1+\kappa)+c\}\{(M_x-M_{x+n}) + p_y(M_{x+1}-M_{x+n})$$

$$+ {}_2p_y(M_{x+2}-M_{x+n}) + \ldots + {}_{n-1}p_y(M_{x+n-1}-M_{x+n})\}\Big]$$

$$= \pi\frac{N_{x-1:y-1}-N_{x+n-1:y+n-1}}{l_y v^{x+n}} - \{\pi(1+\kappa)+c\}\Big(\frac{M_x-M_{x+n}}{v^{x+n}} + p_y\frac{M_{x+1}-M_{x+n}}{v^{x+n}}$$

$$+ \cdots + {}_{n-1}p_y\frac{M_{x+n-1}-M_{x+n}}{v^{x+n}}\Big)$$

which is the reserve by the retrospective method.

21. Mr G. H. Ryan formed for a particular purpose a hypothetical mortality table by adding ·01 to the H^M 3 per cent. P_x at all ages. Show that the policy-values by such a mortality table are necessarily less than the H^M 3 per cent. values.

By this hypothetical table

$$
\begin{aligned}
{}_n\mathrm{V}'_x &= \frac{\mathrm{P}'_{x+n} - \mathrm{P}'_x}{\mathrm{P}'_{x+n} + d} \\
&= \frac{(\mathrm{P}_{x+n} + \cdot 01) - (\mathrm{P}_x + \cdot 01)}{\mathrm{P}_{x+n} + \cdot 01 + d} \\
&= \frac{\mathrm{P}_{x+n} - \mathrm{P}_x}{(\mathrm{P}_{x+n} + d) + \cdot 01}
\end{aligned}
$$

But the true value by $\mathrm{H^M}$ table at 3 per cent.

$$
{}_n\mathrm{V}_x = \frac{\mathrm{P}_{x+n} - \mathrm{P}_x}{\mathrm{P}_{x+n} + d}
$$

Therefore ${}_n\mathrm{V}'_x < {}_n\mathrm{V}_x$ for every value of x and n.

22. Each of $l_{[x]}$ persons effected n years ago at age x a whole-life policy for 1 at an annual premium of $\mathrm{P}_{[x]}$. Find the reserve required to be held to-day in respect of the survivors. How much of this is required for the lives that are still "select," and how much for the now "damaged" lives?

The total reserve required is

$$
l_{[x]+n}(\mathrm{A}_{[x]+n} - \mathrm{P}_{[x]}\mathrm{a}_{[x]+n})
$$

Now of these $l_{[x]+n}$ survivors, $l_{[x+n]}$ are still select. The reserve required for them will accordingly be

$$
l_{[x+n]}(\mathrm{A}_{[x+n]} - \mathrm{P}_{[x]}\mathrm{a}_{[x+n]})
$$

The remainder of the lives are the "damaged," and they number $l_{[x]+n} - l_{[x+n]}$. The reserve required for them will be

$$
\begin{aligned}
&\{v(d_{[x]+n} - d_{[x+n]}) + v^2(d_{[x]+n+1} - d_{[x+n]+1}) + \cdot\ \cdot\ \cdot\ \} \\
&\qquad - \mathrm{P}_{[x]}\{(l_{[x]+n} - l_{[x+n]}) + v(l_{[x]+n+1} - l_{[x+n]+1}) + \cdot\ \cdot\ \cdot\ \} \\
&= (l_{[x]+n} - l_{[x+n]})\frac{(\mathrm{M}_{[x]+n} - \mathrm{M}_{[x+n]}) - \mathrm{P}_{[x]}(\mathrm{N}_{[x]+n} - \mathrm{N}_{[x+n]})}{\mathrm{D}_{[x]+n} - \mathrm{D}_{[x+n]}}
\end{aligned}
$$

The reserve for the "select" lives, together with the reserve for the "damaged," will equal the total reserve required

$$l_{[x+n]}(A_{[x+n]} - P_{[x]}a_{[x+n]}) + (l_{[x]+n} - l_{[x+n]})\frac{(M_{[x]+n} - M_{[x+n]}) - P_{[x]}(N_{[x]+n} - N_{[x+n]})}{D_{[x]+n} - D_{[x+n]}}$$

$$= \frac{M_{[x+n]} - P_{[x]}N_{[x+n]}}{v^{x+n}} + \frac{(M_{[x]+n} - M_{[x+n]}) - P_{[x]}(N_{[x]+n} - N_{[x+n]})}{v^{x+n}}$$

$$= \frac{M_{[x]+n} - P_{[x]}N_{[x]+n}}{v^{x+n}}$$

$$= l_{[x]+n}(A_{[x]+n} - P_{[x]}a_{[x]+n})$$

23. A policy for a term of n years is granted at an annual premium to a life aged x, with the option of continuance at the end of the term as an endowment assurance, maturing at age $(x+n+t)$, on payment of the normal yearly premium at age $(x+n)$. Find expressions for the value of the policy on a select basis, (1) at the end of the $(n-1)$th year, and (2) at the end of the $(n+1)$th year, assuming the option to have been exercised.

The premium payable during the first n years is

$$P = \frac{M_{[x]} - M_{[x]+n}}{N_{[x]} - N_{[x]+n}} + \frac{(P_{[x]+n:\overline{t}|} - P_{[x+n]:\overline{t}|})(N_{[x]+n} - N_{[x]+n+t})}{N_{[x]} - N_{[x]+n}}$$

while the premium thereafter for t years if the option be exercised is $P_{[x+n]:\overline{t}|}$

(1) The reserve at the end of the $(n-1)$th year is

$$\frac{P(N_{[x]} - N_{[x]+n-1}) - (M_{[x]} - M_{[x]+n-1})}{D_{[x]+n-1}}$$

(2) The reserve at the end of the $(n+1)$th year, the option having been exercised, is

$$A_{[x]+n+1:\overline{t-1}|} - P_{[x+n]:\overline{t}|}\,a_{[x]+n+1:\overline{t-1}|}$$

24. If in the formula $\mu_x = A + Bc^x$ the value of c be 1, what effect is produced upon (1) μ_x, (2) ${}^{(\infty)}\overline{P}_x$, and (3) ${}^{(\infty)}_{\ n}\overline{V}_x$?

(1) If $c = 1$, then $\mu_x = A + B$, which is constant for all values of x.

(2) It has already been proved that under such conditions as to mortality

$$\bar{a} = \frac{1}{\mu+\delta} \quad \text{(see page 114)}$$

$$\text{and } \bar{A} = \frac{\mu}{\mu+\delta} \quad \text{(see page 218)}$$

$$\text{Therefore} \quad {}^{(\infty)}\bar{P} = \frac{\frac{\mu}{\mu+\delta}}{\frac{1}{\mu+\delta}}$$

$$= \mu$$

$$(3) \qquad {}^{(\infty)}_{\ n}\bar{V}_x = \bar{A}_{x+n} - {}^{(\infty)}\bar{P}_x \bar{a}_{x+n}$$

$$= \frac{\mu}{\mu+\delta} - \mu\frac{1}{\mu+\delta}$$

$$= 0$$

which is obviously correct, as the premium for each year will meet the year's risk, and there will be no reserve to accumulate.

25. Prove that the expected death-strain under a whole-life policy, subject to an annual premium payable throughout life, increases with the duration of the assurance if, at all ages on the basis of the valuation mortality table and rate of interest, $\Delta^2 a$ is algebraically $> i\Delta a$.

The expected death-strain in the $(n+1)$th year of a policy effected at age x is $q_{x+n}(1 - {}_{n+1}V_x)$, and in the following year $q_{x+n+1}(1 - {}_{n+2}V_x)$. The strain will therefore be increasing, if

$$q_{x+n}(1 - {}_{n+1}V_x) < q_{x+n+1}(1 - {}_{n+2}V_x)$$

$$\text{that is, if} \qquad q_{x+n}\frac{\mathrm{a}_{x+n+1}}{\mathrm{a}_x} < q_{x+n+1}\frac{\mathrm{a}_{x+n+2}}{\mathrm{a}_x}$$

$$\text{if} \qquad \mathrm{a}_{x+n+1} - p_{x+n}\mathrm{a}_{x+n+1} < \mathrm{a}_{x+n+2} - p_{x+n+1}\mathrm{a}_{x+n+2}$$

$$\text{or if} \qquad a_{x+n+1} - (1+i)a_{x+n} < a_{x+n+2} - (1+i)a_{x+n+1}$$

$$\text{if} \quad -(a_{x+n+2} - 2a_{x+n+1} + a_{x+n}) < -i(a_{x+n+1} - a_{x+n})$$

$$\text{that is, if} \qquad \Delta^2 a_{x+n} > i\Delta a_{x+n}$$

which is the condition desired.

26. At the commencement of a certain year a company has on its books l_x persons who have been assured for 1 each for n years, and who will be subject throughout the year to a special rate of mortality $q_x + w_x$, and in respect of the claims which occur amongst them, the company undertakes to pay only the reserve values at the end of the year. On the assumption that the extra mortality will cease at the end of the year, and that it will not prejudicially affect the lives remaining assured, state under what conditions the company will make a profit from the arrangement, and find an expression for the amount of such profit. (Assume the premiums due at the beginning of the year.)

The office will have in hand at the end of the year

$$l_x({}_nV_{x-n} + P_{x-n})(1+i) = l_{x+1} \times {}_{n+1}V_{x-n} + d_x$$

which under normal conditions is sufficient to meet its claims and the reserve values required for the survivors.

Under the conditions stated the number of deaths is increased to $l_x(q_x + w_x) = d_x + w_x l_x$, but the amount paid to each is only ${}_{n+1}V_{x-n}$. Further, the survivors will be $l_x - (d_x + w_x l_x) = l_{x+1} - w_x l_x$. Therefore to meet its actual claims and the reserves for the survivors the office should have in hand at the end of the year

$$\begin{aligned}(l_{x+1} - w_x l_x)_{n+1}V_{x-n} &+ (d_x + w_x l_x)_{n+1}V_{x-n} \\ &= l_{x+1} \times {}_{n+1}V_{x-n} + d_x \times {}_{n+1}V_{x-n}\end{aligned}$$

The office will therefore make a profit, if

$$l_{x+1} \times {}_{n+1}V_{x-n} + d_x > l_{x+1} \times {}_{n+1}V_{x-n} + d_x \times {}_{n+1}V_{x-n}$$

that is, if $\qquad d_x(1 - {}_{n+1}V_{x-n}) > 0$

which must always happen, as ${}_{n+1}V_{x-n}$ is fractional.

Further, the amount of the profit must be $d_x(1 - {}_{n+1}V_{x-n})$.

27. The actual claims for the year in an office exceed the expected amount. Does the difference represent the loss from mortality during the year? Give the reasons for your answer.

This difference does not represent loss from mortality. The office has in hand for each policy on its books a reserve value of certain amount, greater or smaller. A comparison merely between

actual and expected claims cannot therefore indicate the profit or loss from mortality. The loss to the office through the death of any single life assured is not the sum assured, as would be implied in such a comparison as that suggested, but the difference between the sum assured and the above-mentioned reserve. It is this difference, then, that must be taken account of, if we are to make a true investigation into the profit or loss from mortality.

The reserve held by the office at the beginning of the year and the net premium then paid, both accumulated to the end of the year, provide two things:—

(1) The reserve value required at the end of the year; and

(2) A contribution towards mortality risk.

The total of the contributions to mortality during the year would require to be compared with the net loss to the office in respect of the claims, *i.e.*, the difference between the sums assured and the reserve values as described above, the balance between these totals representing profit or loss from mortality.

It is indeed conceivable that the actual claims in an office might exceed the expected, and nevertheless a profit from mortality result. For the claims might have occurred chiefly among old assured lives where the reserve values were considerable, and the actual loss to the office consequently small.

On the other hand, the actual claims might be well within the expected, and yet there might be a loss from mortality, due to the fact that the claims occurred chiefly amongst recently assured lives where the reserves in hand were small.

In this connection it may be pointed out that in the annual reports issued by insurance companies, one may sometimes observe a table giving the distribution of the claims experienced according to the age attained at death. Such a table, however, conveys but little information regarding the mortality experience of the company. It is obvious that much would depend on a variety of considerations, such as the average age at entry and the class of insurance effected. A company transacting little whole-life business, and that chiefly at the younger ages at entry, but doing a large endowment assurance business, would tend to compare unfavourably in such a comparison with another office doing very little endowment assurance business, and a considerable amount of whole-life business chiefly at the older ages at entry. We could not conclude merely for this reason that the profit from mortality in the first company is less than in the second.

28. Assuming that an office had on its books at the commencement of a year a group of 1000 lives aged 40, each of whom was insured under a policy for £100 (without profits) payable at age 55 or death, and effected exactly 10 years previously at an annual premium of £3, 14s.; also assuming that 10 of these become claims (payable at the end of the year of death) during the year, the remainder being still in force at the end of the year; that the office earns 4 per cent. interest on its funds, spends 10 per cent. of its premiums, and makes an H^M 3 per cent net valuation; find the total profit to the office earned by the group during the year. How much of this is (1) profit from mortality; (2) profit from interest; (3) profit from loading?

From Hardy's "Valuation Tables" we find

$$100\,{}_{10}V_{30:\overline{25}|} = 100A_{40:\overline{15}|} - 100P_{30:\overline{25}|}a_{40:\overline{15}|}$$
$$= 66{\cdot}847 - 3{\cdot}244 \times 11{\cdot}383$$
$$= 29{\cdot}921$$
$$100\,{}_{11}V_{30:\overline{25}|} = 100A_{41:\overline{14}|} - 100P_{30:\overline{25}|}a_{41:\overline{14}|}$$
$$= 68{\cdot}528 - 3{\cdot}244 \times 10{\cdot}805$$
$$= 33{\cdot}477$$

The accumulation of the fund is as follows:—

Fund at beginning of year, 1000 × 29·921 . . .		29921
Add—Office premiums paid, 1000 × 3·7 . .	3700	
Less—10 per cent. for expenses . .	370	
		3330
		33251
Add also one year's interest thereon at 4 per cent., the rate realised		1330
		34581
Deduct the claims payable		1000
Actual fund at end of year		33581
From which deduct the office's liability, 990 × 33·477 .		33142
Difference, being total profit earned by the group during the year		439

Total profit, as before			439
Made up thus :—			
(1) The liability at beginning of year was .		29921	
H^M 3 per cent. net premiums, $1000 \times 3 \cdot 244$		3244	
		33165	
Interest for year at 3 per cent., the valuation rate		995	
		34160	
Less—Claims payable as before . . .		1000	
		33160	
Deduct—Liability at end of year as before .		33142	
Difference, being profit from mortality .		18	
(2) The office premiums were as before .	3700		
The net premiums	3244		
Difference, being loading . .	456		
Less—Expenses as before . .	370		
	86		
Add—Interest for year at valuation rate	3		
Profit from loading		89	
(3) The interest realised was as before .	1330		
Deduct—Interest already taken credit for in heading (1) . . 995			
Do. heading (2) . . 3	998		
Difference, being profit from interest .		332	
			439

29. What is meant by an H^M and $H^{M(5)}$ valuation? How do the reserves by this basis compare with those required by Dr Sprague's Select Tables?

For the purposes of an H^M and $H^{M(5)}$ valuation, in the fundamental formula

$$ {}_nV_x = A_{x+n} - P_x(1 + a_{x+n}) $$

we insert values for the functions on the right-hand side of the equation, as follows: P_x,—the H^M net premiums are used throughout; A_{x+n} and a_{x+n},—when the policy is of less than five years' duration, the values from the H^M table are used, but when five or more years have elapsed, the $H^{M(5)}$ values are taken.

In the case of policies, which have been less than five years in force, the H^M reserve is less than that required by Dr Sprague's Select Tables at all ages at entry. For those policies of five or more than five years' duration we may compare the formulas, as follows:—

$$ {}_nV_x^{(H^M \text{ and } H^{M(5)})} = A_{x+n}^{(H^{M(5)})} - P_x^{(H^M)} a_{x+n}^{(H^{M(5)})} $$

$$ \text{and} \quad {}_nV_{[x]} = A_{x+n}^{(H^{M(5)})} - P_{[x]} a_{x+n}^{(H^{M(5)})} $$

It will be seen that the only difference between them is that the H^M and $H^{M(5)}$ valuation employs the H^M net premium, while under the other the select premium is used. Now $P_x^{(H^M)}$ is less than $P_{[x]}$ of Dr Sprague's Tables for all ages at entry up to 43, and thereafter is greater. Hence the H^M and $H^{M(5)}$ reserve is greater than the select reserve for ages at entry up to 43, and thereafter is less.

It may be mentioned that for an average office an H^M and $H^{M(5)}$ valuation gives a close approximation over the whole business to the reserve required by Dr Sprague's Select Tables.

CHAPTER XIX

Life Interests and Reversions

1. In practice it will be found that Life Interests are usually payable with a proportion to date of death, and therefore in such a case, though *Text Book* formula (1) is commonly used, a more exact formula would be

$$\begin{aligned} \bar{a}'_x &= a'_x + \tfrac{1}{2}A'_x \\ &= \frac{v - P'_x}{P'_x + d} + \frac{1}{2}\frac{P'_x}{P'_x + d} \\ &= \frac{v - \dfrac{P'_x}{2}}{P'_x + d} \end{aligned}$$

A policy must then be taken out for $\dfrac{1 - \dfrac{d}{2}}{P'_x + d}$, and we have

Amount paid to vendor	$\dfrac{v - \dfrac{P'_x}{2}}{P'_x + d}$
First premium on policy	$\dfrac{P'_x\left(1 - \dfrac{d}{2}\right)}{P'_x + d}$
Total outlay	$\dfrac{v\left(1 + \dfrac{P'_x}{2}\right)}{P'_x + d}$

The annual income is		1
to be applied in payment of		
One year's interest on total outlay .	$\dfrac{d\left(1+\frac{P'_x}{2}\right)}{P'_x+d}$	
Annual premium on policy . .	$\dfrac{P'_x\left(1-\frac{d}{2}\right)}{P'_x+d}$	
		1
At death there is received—		
Sum assured under policy		$\dfrac{1-\frac{d}{2}}{P'_x+d}$
Proportionate payment of life interest . . .		$\frac{1}{2}$
Together		$\dfrac{1+\frac{P'_x}{2}}{P'_x+d}$
which is to be applied in payment of		
Total outlay	$\dfrac{v\left(1+\frac{P'_x}{2}\right)}{P'_x+d}$	
One year's interest thereon . .	$\dfrac{d\left(1+\frac{P'_x}{2}\right)}{P'_x+d}$	
		$\dfrac{1+\frac{P'_x}{2}}{P'_x+d}$

2. Where the life interest to (x) is limited to n years, the formula is on the same lines as *Text Book* formula (1). The policy to be effected is an endowment assurance which will return the total outlay with a year's interest at the end of the first year in which no payment of life interest is made, that is, at the end of the $(n+1)$th year or the year of previous death; and we shall have

$$a'_{\overline{n}|} = \frac{1}{P'_{x:\overline{n+1}|}+d} - 1$$

8. The problem as stated in *Text Book*, Article 13, is much more frequent in practice than the problem to find the value of the life interest, and therefore it will be well to state the matter from that point of view.

Since $\frac{1}{P'_x+d}-1$ purchases a life interest of 1, a sum of 1 will purchase a life interest of $\frac{P'_x+d}{1-(P'_x+d)}$, and the policy to be effected is for $\frac{1}{1-(P'_x+d)}$. Then we have

Amount paid to vendor		1
First premium on policy		$\frac{P'_x}{1-(P'_x+d)}$
Total outlay		$\frac{v}{1-(P'_x+d)}$
The annual income is		$\frac{P'_x+d}{1-(P'_x+d)}$
which provides for		
One year's interest on total outlay .	$\frac{d}{1-(P'_x+d)}$	
Annual premium on policy . . .	$\frac{P'_x}{1-(P'_x+d)}$	
		$\frac{P'_x+d}{1-(P'_x+d)}$
At death the sum assured is received . . .		$\frac{1}{1-(P'_x+d)}$
which repays		
The total outlay	$\frac{v}{1-(P'_x+d)}$	
And one year's interest thereon . .	$\frac{d}{1-(P'_x+d)}$	
		$\frac{1}{1-(P'_x+d)}$

4. In connection with the Reversionary Life Interest, formula (2), as mentioned in *Text Book*, Article 51, was given by Mr Charles Jellicoe. It is assumed under it that an annuity for the joint lives is actually purchased or set up in the books of the office. Dr Sprague, arguing that if the number of contracts entered into is sufficiently large, no such procedure is required, or, as a matter of fact, carried out, suggested formula (5), where one rate of interest is assumed throughout, and which, without the correction for $\frac{1}{2}$ payable if (y) die first, reads

$$a'_{y|x} = \frac{1-(P'_x+d_{(i)})(1+a_{xy(i)})}{P'_x+d_{(i)}}$$

Or, if we are to assume a higher rate of interest till the life interest comes into possession

$$a'_{y|x} = \frac{1-(P'_x+d_{(i')})(1+a_{xy(i')})}{P'_x+d_{(i)}}$$

For a complete reversionary life interest we might use the formula

$$\begin{aligned}\mathring{a}'_{y|x} &= \mathring{a}'_x - \mathring{a}_{xy} \\ &= \frac{v-\dfrac{P'_x}{2}}{P'_x+d} - \mathring{a}_{xy} \\ &= \left(1-\frac{d}{2}\right)\frac{1-(P'_x+d)(1+a_{xy})}{P'_x+d} \\ &= v^{\frac{1}{2}}\,\frac{1-(P'_x+d)(1+a_{xy})}{P'_x+d}\end{aligned}$$

which agree with formulas (16) and (17) of *Text Book*, Chapter XIV.

If now we give $\dfrac{1}{P'_x+d}-(1+a_{xy})$ for an annual reversionary charge of 1, we shall give 1 for a similar charge of $\dfrac{P'_x+d}{1-(P'_x+d)(1+a_{xy})}$. In this case a policy must be effected for $\dfrac{1}{1-(P'_x+d)(1+a_{xy})}$. Proceeding as before, we have

Amount paid to vendor	1
First premium on policy . . .	$\dfrac{P'_x}{1-(P'_x+d)(1+a_{xy})}$
Price of joint-life annuity . . .	$\dfrac{P'_x+d}{1-(P'_x+d)(1+a_{xy})}a_{xy}$
Total outlay	$\dfrac{v}{1-(P'_x+d)(1+a_{xy})}$

The annual income, from the annuity during the joint lives and from the life interest after (y)'s death, is . $\dfrac{P'_x+d}{1-(P'_x+d)(1+a_{xy})}$

which provides for

One year's interest on total outlay .	$\dfrac{d}{1-(P'_x+d)(1+a_{xy})}$	
Annual premium on policy . .	$\dfrac{P'_x}{1-(P'_x+d)(1+a_{xy})}$	
		$\dfrac{P'_x+d}{1-(P'_x+d)(1+a_{xy})}$

At (x)'s death the sum assured is received $\dfrac{1}{1-(P'_x+d)(1+a_{xy})}$

which repays

The total outlay .	$\dfrac{v}{1-(P'_x+d)(1+a_{xy})}$	
And one year's interest thereon .	$\dfrac{d}{1-(P'_x+d)(1+a_{xy})}$	
		$\dfrac{1}{1-(P'_x+d)(1+a_{xy})}$

5. For the Absolute Reversion, Jellicoe's formula (8) assumes as before that an annuity is actually or constructively purchased, while Sprague's formula (9) rejects this as not in accordance with customary practice, and adopts one rate of interest throughout, assuming that reversions will be purchased in sufficient numbers to warrant this procedure. It is true that a specially large reversion

might throw out the average on which the latter argument rests, but this merely indicates that the office must avoid contracts of such size, just as they have a limit for the amount of assurance on any one life.

As before, let us consider the reversion which can be purchased for a given sum, as that is the more usual problem. A reversion of 1 is purchased for $1-d(1+a_x)$; therefore 1 will purchase a reversion of $\frac{1}{1-d(1+a_x)}$. And an annuity of $\frac{d}{1-d(1+a_x)}$ must be purchased. Then

Amount paid to vendor	1
Price of annuity	$\frac{d}{1-d(1+a_x)}a_x$
Total outlay	$\frac{v}{1-d(1+a_x)}$
The annual income from the annuity is . .	$\frac{d}{1-d(1+a_x)}$

which is interest on the total outlay.

The amount received at (x)'s death is . . .		$\frac{1}{1-d(1+a_x)}$
which repays		
Total outlay	$\frac{v}{1-d(1+a_x)}$	
One year's interest thereon . .	$\frac{d}{1-d(1+a_x)}$	
		$\frac{1}{1-d(1+a_x)}$

6. In the case of the Contingent Reversion, Jellicoe's formula (10) proceeds on the same principle as his others, viz., of purchasing an annuity during the joint lives; while Sprague's, formula (12), has one rate throughout. Sprague's formula cannot, however, be further reduced to $A_{\overset{1}{xy}(t)}$, for it is necessary, as in the immediate and reversionary life interests, to set up a policy to cover possible loss of capital which will happen here if (x) dies before (y). In the absolute reversion the only possible loss was that of interest so long as the life lived, and accordingly no policy to cover loss of capital was necessary.

Now if $1-(P'^{1}_{xy}+d)(1+a_{xy})$ purchases a contingent reversion of 1, then 1 will purchase a contingent reversion of $\dfrac{1}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$, for which amount a policy on (x) against (y) must be effected. Also an annuity of $\dfrac{P'^{1}_{xy}+d}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$ must be purchased. Accordingly we have

Amount paid to vendor . . .	1
First premium on policy . . .	$\dfrac{P'^{1}_{xy}}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$
Price of joint-life annuity . . .	$\dfrac{P'^{1}_{xy}+d}{1-(P'^{1}_{xy}+d)(1+a_{xy})}a_{xy}$
Total outlay	$\dfrac{v}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$

The annual income from the joint-life annuity is $\dfrac{P'^{1}_{xy}+d}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$

which will provide

One year's interest on total outlay . .	$\dfrac{d}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$	
Annual premium on policy . . .	$\dfrac{P'^{1}_{xy}}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$	
		$\dfrac{P'^{1}_{xy}+d}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$

If (x) dies first, the sum assured by the policy falls in, and if (y) dies first, the amount in reversion, amounting to . . . $\dfrac{1}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$

which gives

Total outlay . .	$\dfrac{v}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$	
One year's interest thereon . .	$\dfrac{d}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$	
		$\dfrac{1}{1-(P'^{1}_{xy}+d)(1+a_{xy})}$

7. The method of book-keeping to be used in connection with reversions presents some difficulties. There are three common methods to be chosen from.

(1) There is the theoretical method, upon which Jellicoe's formulas proceed, of actually purchasing from the annuity department an annuity to pay premium and interest or interest alone, as the case may be. The amount available to purchase these annuities is the difference between the sums in reversion and their present value: it must be added to the amount invested in reversions, and will there appear as an asset, while it will also be entered as a liability by being included in the annuity fund. The annuity, as explained, will always provide the necessary annual income. If, however, Sprague's formulas be used and this method followed, the annuity will have to be purchased on, say, a 5 per cent. basis, which of course is not profitable to the annuity department.

(2) Interest at an assumed rate may be added yearly to the value of the reversion, and taken credit for in the revenue account. But the annual increase in the value of the reversion is not so great as a year's interest, and if only a few reversions were to fall in during a quinquennium a large sum might have to be written off the reversions account at the valuation. Where the business is new or of no great extent this might easily occur, and accordingly in such circumstances this method is not to be recommended.

(3) The safest method is to add only the yearly increase in the value of the reversion, and take credit for the whole difference between the fund in reversion and its value when it falls in. Such a stringent method means, of course, that this form of investment may show at one time a very low rate of interest, and at another a rate swollen out of all proportion. But obviously a loss cannot be incurred at any time.

EXAMPLES

1. Determine the present value of an annuity-certain of £1 per annum for n years, which is to pay during its continuance a given rate of interest on the purchase money, and to replace the purchase money at the expiration of the term at a different rate of interest.

On the principles of the life interest,

$$a'_{\overline{n}|} = \frac{1}{P'_{\overline{n+1}|} + d} - 1$$

where d is at the rate of interest to be realised, and $P'_{\overline{n+1|}}$ is the sinking fund payable in advance required to amount to 1 at the end of $(n+1)$ years, calculated at the rate of interest for replacing.

2. A sum of £10,000 is required on the security of an ample reversionary life interest to (35) after (65). It is required to find what annual charge on the life interest must be given, and for what sum (35) must be assured, the office premium on his life being £2, 8s. 11d. per cent., and interest being taken at 5 per cent., with the joint-life annuity based on the Carlisle Table.

The annual charge will be

$$\frac{10000(P'_{35}+d)}{1-(P'_{35}+d)(1+a_{35:65})} = \frac{10000(\cdot 02446+\cdot 04762)}{1-(\cdot 02446+\cdot 04762)8\cdot 143}$$
$$= 1745$$

and the amount of the policy

$$\frac{10000}{1-\cdot 58695} = 24210.$$

3. What charge must be placed on funds, over which an absolute reversion is held payable at the death of a person aged 65, in consideration of a sum down of £10,000? The Carlisle Table with 5 per cent. interest is to be used.

The charge is

$$\frac{10000}{A_{65}} = \frac{10000}{\cdot 58262}$$
$$= 17164.$$

4. A sum of £20,000 is desired, to be secured over funds falling to a person aged 30 should he be alive at the death of his mother aged 60. Find the amount of the necessary charge on the funds, having given that $P'^{\,1}_{30:60} = \cdot 01354$. The Carlisle Table is to be used for the joint-life annuity, and 5 per cent. interest is to be assumed.

The charge is

$$\frac{20000}{1-(P'^{\,1}_{30:60}+d)(1+a_{30:60})} = \frac{20000}{1-(\cdot 01354+\cdot 04762)9\cdot 196}$$
$$= 45707.$$

CHAPTER XX

Sickness Benefits

EXAMPLES

1. If the law of sickness be such that at any age two are constantly sick for one that dies, find the single premium for a sickness allowance of 10s. a week at age x, to cease at age 65.

Out of l_x persons alive at age x the number constantly sick during the first year is $2d_x$, during the second $2d_{x+1}$, etc. Therefore the amount paid in sickness allowance is $26 \times 2d_x$, $26 \times 2d_{x+1}$, etc., and the present value of the payments is

$$v^{\frac{1}{2}}52(d_x + vd_{x+1} + \cdot \cdot \cdot + v^{64-x}d_{64})$$

and the value of the benefit to each person is

$$52(1+i)^{\frac{1}{2}} \frac{C_x + C_{x+1} + \cdot \cdot \cdot + C_{64}}{D_x}$$

$$= 52(1+i)^{\frac{1}{2}} \frac{M_x - M_{65}}{D_x}$$

2. Find the weekly premium required at age x to provide the following benefits :—

(*a*) £25 at death.

(*b*) A sickness allowance of 10s. per week limited to n years.

(*c*) A deferred annuity of 10s. per week to be entered upon at the end of n years.

$$\text{Benefit side} \quad = 25\frac{M_x(1+i)^{\frac{1}{2}}}{D_x} + \cdot 5\frac{K_x - K_{x+n}}{D_x} + 26\frac{N_{x+n} + \frac{1}{2}D_{x+n}}{D_x}$$

$$\text{Payment side} = P\frac{(N_x - N_{x+n}) + \frac{1}{2}(D_x - D_{x+n})}{D_x}$$

P being the annual contribution.

Equating and solving,

$$P = \frac{25M_x(1+i)^{\frac{1}{2}} + \cdot 5(K_x - K_{x+n}) + 26(N_{x+n} + \frac{1}{2}D_{x+n})}{N_x - N_{x+n} + \frac{1}{2}(D_x - D_{x+n})}$$

And the weekly premium is $\frac{P}{52}$.

3. Find the weekly contribution required at age x to provide the following benefits:—

(*a*) £20 at death.

(*b*) A sickness allowance of £1 per week for the first six months' sickness, 10s. per week for the second six months, and 5s. per week thereafter, the whole benefit to cease at the end of n years.

(*c*) A deferred annuity of 5s. per week to be entered upon at the end of n years.

Benefit side

$$= 20\frac{M_x(1+i)^{\frac{1}{2}}}{D_x} + \frac{(K^{I}_x - K^{I}_{x+n}) + \cdot 5(K^{II}_x - K^{II}_{x+n}) + \cdot 25(K^{III}_x - K^{III}_{x+n})}{D_x}$$

$$+ 13\frac{N_{x+n} + \frac{1}{2}D_{x+n}}{D_x}$$

$$\text{Payment side} = P\frac{(N_x - N_{x+n}) + \frac{1}{2}(D_x - D_{x+n})}{D_x}.$$

Hence P may be found and the weekly contribution is $\frac{P}{52}$.

CHAPTER XXI

Construction of Tables

1. A table of $\log D_x$ formed as shown in *Text Book*, Article 49, at rate i may be checked very simply with the table at rate j.

For at rate i,

$$\Sigma_x^{\omega-1} \log D_x = (\log l_x + \log l_{x+1} + \cdots + \log l_{\omega-1}) + \log v(x + \overline{x+1} + \cdots + \overline{\omega-1})$$

And at rate j,

$$\Sigma_x^{\omega-1} \log D_x = (\log l_x + \log l_{x+1} + \cdots + \log l_{\omega-1}) + \log v'(x + \overline{x+1} + \cdots + \overline{\omega-1})$$

Therefore

$$\Sigma_x^{\omega-1} \log D_x \text{ at rate } j = \Sigma_x^{\omega-1} \log D_x \text{ at rate } i + (\log v' - \log v)\,\frac{(\omega-x)(\omega+x-1)}{2}$$

2. A table of A_x may be formed with the help of Gauss's logarithms in a way similar to that in which $A^{\,1}_{xy}$ is formed as shown in *Text Book*, Article 99.

$$\begin{aligned} A_x &= vq_x + vp_x A_{x+1} \\ &= vp_x\left(\frac{1}{p_x} - 1\right) + vp_x A_{x+1} \\ &= vp_x(\Pi_x + A_{x+1}) \end{aligned}$$

where $\Pi_x = \dfrac{1}{p_x} - 1$

Hence $\log A_x = \log vp_x + \log \Pi_x + [t](\log A_{x+1} - \log \Pi_x)$

Starting then at the end of the table we have $A_{\omega-1} = v$ and $\log A_{\omega-1} = \log v$. From $\log v$ deduct $\log \Pi_{\omega-2}$ as tabulated (for the *Text Book* table at pages 499 and 501); enter Gauss's table with the difference as argument, and to the result add

$\log \Pi_{\omega-2}$ and $\log vp_{\omega-2}$, and we have $\log A_{\omega-2}$. From $\log A_{\omega-2}$ deduct $\log \Pi_{\omega-3}$; enter Gauss's table, and to the result add $\log \Pi_{\omega-3}$ and $\log vp_{\omega-3}$, and we have $\log A_{\omega-3}$, and so on, to age 0. Then take the antilogs, and the table of A_x is formed. The table of A_x when formed may be simply checked with the table of a_x. For

$$A_x + A_{x+1} + \cdots + A_{\omega-1} = (v - da_x) + (v - da_{x+1}) + \cdots + (v - da_{\omega-1})$$
$$= (\omega - x)v - d(a_x + a_{x+1} + \cdots + a_{\omega-1})$$

3. Besides the method of tabulating P_x given in *Text Book,* Article 56, we might enter annual-premium conversion tables with a_x, and so obtain P_x, as described in Chapter VIII. Or again, we might make use of a table of reciprocals, which we should enter with $1 + a$, and from the result deduct d. Thus—

Age (1)	a (2)	$\frac{1}{1+a}$ (3)	$P = \frac{1}{1+a} - d$ (4)

Neither of these methods, however, is a continued method.

4. The arithmometer may be employed to form a table of $A_{\overline{xn}|}$ in the same way as described in *Text Book,* Article 61, for A_x.

A preliminary table of the differences between the temporary annuities must first be drawn up, thus:—

1. $a_{x:\overline{n-1}|} - a_{x+1:\overline{n-2}|}$
2. $a_{x+1:\overline{n-2}|} - a_{x+2:\overline{n-3}|}$
3. $a_{x+2:\overline{n-3}|} - a_{x+3:\overline{n-4}|}$

etc.

where the age at which the annuity ceases is always the same, viz., $x+n-1$.

Putting 1 on the slide and d on the fixed plate with the regulator at subtraction, multiply d by $(1+a_{x:\overline{n-1}|})$ and the value of $A_{\overline{xn}|}$ will result. On changing the regulator to addition, continued multiplication of d by the series of differences found as above, will give the values of $A_{x+1:\overline{n-1}|}$, $A_{x+2:\overline{n-2}|}$, etc. For

$$A_{x+1:\overline{n-1}|} = 1-d(1+a_{x+1:\overline{n-2}|}) = A_{\overline{xn}|}+d(a_{x:\overline{n-1}|}-a_{x+1:\overline{n-2}|})$$

$$A_{x+2:\overline{n-2}|} = A_{x+1:\overline{n-1}|}+d(a_{x+1:\overline{n-2}|}-a_{x+2:\overline{n-3}|})$$

etc. etc.

5. Instead of using the values of $-\Delta a_x$ to help in forming the table of policy-values as described in *Text Book*, Article 78, we may use the annuity-due values themselves.

For $$_{n}V_{x} = 1-\frac{\text{a}_{x+n}}{\text{a}_{x}}$$

Therefore, putting $\frac{1}{\text{a}_x}$ on the fixed plate, multiplying successively by a_{x+1}, a_{x+2}, etc., and using the "effacer" between each operation, we get the values $\frac{\text{a}_{x+1}}{\text{a}_x}$, $\frac{\text{a}_{x+2}}{\text{a}_x}$, etc., the complements of which are the required policy-values.

6. The values of endowment assurance policies may be similarly arrived at, since

$$_{n}V_{\overline{xr}|} = 1-\frac{\text{a}_{x+n:\overline{r-n}|}}{\text{a}_{\overline{xr}|}}$$

Or they may be formed on the principles of *Text Book*, Article 78, since

$$_{n+1}V_{\overline{xr}|} = {}_{n}V_{\overline{xr}|}+\frac{a_{x+n:\overline{r-n-1}|}-a_{x+n+1:\overline{r-n-2}|}}{1+a_{x:\overline{r-1}|}}$$

A preliminary table, as for the tabulating of $A_{\overline{xn}|}$, must therefore be formed, consisting of

$$a_{x:\overline{r-1}|} - a_{x+1:\overline{r-2}|}$$
$$a_{x+1:\overline{r-2}|} - a_{x+2:\overline{r-3}|}$$
$$a_{x+2:\overline{r-3}|} - a_{x+3:\overline{r-4}|}$$
etc.

Then with the regulator at addition, and $\dfrac{1}{1+a_{x:\overline{r-1}|}}$ on the fixed plate, the successive multiplication by these differences will give us ${}_1V_{\overline{xr}|}$, ${}_2V_{\overline{xr}|}$, ${}_3V_{\overline{xr}|}$, etc.

The results may be checked by addition for

$${}_1V_{\overline{xr}|} + {}_2V_{\overline{xr}|} + \cdots + {}_{r-1}V_{\overline{xr}|}$$
$$= \frac{a_{x:\overline{r-1}|} - a_{x+1:\overline{r-2}|}}{1+a_{x:\overline{r-1}|}} + \frac{a_{x:\overline{r-1}|} - a_{x+2:\overline{r-3}|}}{1+a_{x:\overline{r-1}|}} + \cdots + \frac{a_{x:\overline{r-1}|}}{1+a_{x:\overline{r-1}|}}$$
$$= \frac{(r-1)a_{x:\overline{r-1}|} - (a_{x+1:\overline{r-2}|} + a_{x+2:\overline{r-3}|} + \cdots + a_{x+r-2:\overline{1}|})}{1+a_{x:\overline{r-1}|}}$$

7. The construction of tables of policy-values for limited-payment policies is a slower process, as the premiums have to be valued separately from the sums assured and the difference taken.

As a preliminary, a table of differences of annuity-values should be formed, as in the case of endowment assurances.

Years in force.	Annuity Δ
$n-2$	$a_{x+n-2:\overline{1}|}$
$n-3$	$a_{x+n-3:\overline{2}|} - a_{x+n-2:\overline{1}|}$
$n-4$	$a_{x+n-4:\overline{3}|} - a_{x+n-3:\overline{2}|}$
⋮	⋮
2	$a_{x+2:\overline{n-3}|} - a_{x+3:\overline{n-4}|}$
1	$a_{x+1:\overline{n-2}|} - a_{x+2:\overline{n-3}|}$

Putting ${}_n\mathrm{P}_x$ on the fixed plate, and multiplying by 1, we get the value of the premiums outstanding at the beginning of the last year of premium payment, *i.e.* ${}_n\mathrm{P}_x$. Then the successive multiplication of ${}_n\mathrm{P}_x$ by the quantities found above and their continued addition will give the value of the premiums outstanding at the beginning of each year down to the second. For

$$
\begin{aligned}
{}_n\mathrm{P}_x \mathbf{a}_{x+n-2:\overline{2}|} &= {}_n\mathrm{P}_x + {}_n\mathrm{P}_x a_{x+n-2:\overline{1}|} \\
{}_n\mathrm{P}_x \mathbf{a}_{x+n-3:\overline{3}|} &= {}_n\mathrm{P}_x \mathbf{a}_{x+n-2:\overline{2}|} + {}_n\mathrm{P}_x (a_{x+n-3:\overline{2}|} - a_{x+n-2:\overline{1}|}) \\
\text{etc.} \qquad & \qquad \text{etc.}
\end{aligned}
$$

The results may be checked by addition, since the total

$$
\begin{aligned}
&= {}_n\mathrm{P}_x\{(1 + a_{x+1:\overline{n-2}|}) + (1 + a_{x+2:\overline{n-3}|}) + (1 + a_{x+3:\overline{n-4}|}) + \cdots \\
&\qquad + (1 + a_{x+n-2:\overline{1}|}) + 1\} \\
&= {}_n\mathrm{P}_x\{(n-1) + (a_{x+1:\overline{n-2}|} + a_{x+2:\overline{n-3}|} + a_{x+3:\overline{n-4}|} + \cdots \\
&\qquad + a_{x+n-2:\overline{1}|})\}
\end{aligned}
$$

The value of the premiums must be deducted from the corresponding assurance value to get the value of the policy. The total may be checked by addition, for it should be equal to $(\mathrm{A}_{x+1} + \mathrm{A}_{x+2} + \cdots + \mathrm{A}_{x+n-1})$, less the above summation of the values of the premiums.

The value of the policy after the premiums are paid up is, of course, just the assurance value.

EXAMPLES

1. Show in detail how to obtain a table of annual premiums for whole-life assurances from the values of q_x without constructing the life table. Assuming a rate of mortality represented by a constant addition of ·01 to q_x according to a standard table, explain how the required premiums could be approximately obtained without special tables.

Write down in a column in reverse order the values of p_x from age $\omega - 2$ downwards. From these values prepare a column of $\log vp_x$.

Then

$$\begin{aligned}\log vp_{\omega-2} &= \log a_{\omega-2}\\ \log vp_{\omega-3}+[t]\log a_{\omega-2} &= \log a_{\omega-3}\\ \log vp_{\omega-4}+[t]\log a_{\omega-3} &= \log a_{\omega-4}\\ \text{etc.} &\qquad \text{etc.}\end{aligned}$$

From this last column pass to the values of $a_{\omega-2}$, $a_{\omega-3}$, etc. Enter annual-premium conversion tables with these values, and obtain $P_{\omega-2}$, $P_{\omega-3}$, etc. The following schedule exhibits the process :—

Age x (1)	q_x (2)	$1-(2)$ $=p_x$ (3)	$\log(3)$ $+\log v$ $=\log vp_x$ (4)	$\log(4)$ $+[t]\log a_{x+1}$ $=\log a_x$ (5)	$\log^{-1}(5)$ $=a_x$ (6)	$\frac{1}{1+(6)}-d$ $=P_x$ (7)
$\omega-2$						
$\omega-3$						
$\omega-4$						
etc.						

With reference to the second part of the question it was shown on page 232 that it may be reasonably assumed that the addition of a constant ·01 to the rate of mortality will have the same effect as an increase of ·01 in the rate of interest per unit. We may examine this assumption with reference to an increase from 3 per cent. to 4 per cent. in the rate of interest employed in annuity values. The assumption is that in any table

$$\begin{aligned}v_{(4\%)}p_x &= v_{(3\%)}p'_x\\ \frac{1}{1\cdot04}p_x &= \frac{1}{1\cdot03}p'_x\\ p'_x &= \frac{1\cdot03}{1\cdot04}p_x\\ 1-q'_x &= \frac{1\cdot03}{1\cdot04}(1-q_x)\\ q'_x &= \frac{1\cdot03}{1\cdot04}q_x+\frac{\cdot01}{1\cdot04}\\ &= q_x+\cdot01 \text{ approximately.}\end{aligned}$$